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The Concept of Multiset Category

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Abstract

In this paper category of multisets (**Mul**) is presented and various operations of category of sets such as points, products, sums, etc., were introduced in **Mul** and related results were proved. It is further shown that the product of two objects is unique as is the case for the sum, and the terminal object helps in separating arbitrary arrows. Thus, similar to category of sets if two arrows *agree* on points, they are the same arrows.

Keywords: Set; multiset; category; bimorphism; isomorphism; product; sum.

Mathematics Subject Classification 2010: 18A05, 18A20, 18B05, 18A99.

1 Introduction

Categories were first introduced in the course of formulating algebraic topology, specifically with Samuel Eilenberg's observation that Saunders Maclane's calculations on a specific case of a group extension coincided precisely with Norman Steenrod's calculation of the homology of a *solenoid* (see [1] for details). Eilenberg and Maclane's effort to make sense of this coincidence across apparently distinct areas of mathematical inquiry gave rise to introducing *category theory*. The central notion at work was that of *natural transformations*. In order to provide a broad mathematical perspective, the notion of *functor* was introduced for which they borrowed the term *category* from the philosophical writings of Aristotle, Kant and reiterated in C. S. Peirce (see [2] for details).

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Emmy Noether (one of Maclane's teachers), in formalizing abstract processes, realized that understanding of a mathematical structure in its proper perspective could be better achieved through a proper understanding of the processes preserving that structure. Maclane and Eilenberg proposed an axiomatic formalization of the relation between structures and the processes preserving them, which is considered as a first sustained formalization of Noether's intuitive notion of the concept of category (see [2] for details).

Category theory is based on the idea that the structure of a class of mathematical objects can be best understood by analyzing (instead of the objects themselves) the functions between them. Consider as an example, groups. A group is a set of elements endowed with a binary operation o, a urinary operation $-^1$ (inversing) and a special element ℓ such that the well known properties are satisfied. If we take a set of elements S, we can consider it either as a set or as a group, depending on whether we assume that the above mentioned operations are defined or not.

Category theory takes different view of the problem. Let $\boldsymbol{G}$ be the class of all groups. A mapping $f: \boldsymbol{G} \rightarrow \boldsymbol{G}$ is a function between elements of $\boldsymbol{G}$ considered as sets with the additional requirement that the following properties be satisfied.

$f(a\ o\ b) = f(a)\ o\ f(b), f(\ell) = \ell, f(a-^1) = f(a)-^1$. This characterization of the functions that transform a group in to another is enough to characterize the structure of the groups themselves; a group is simply seen as a set belonging to a class for which functions like this are the morphisms. This example shows the basic idea of category theory; the structure of a certain class of a mathematical object can be studied conveniently by studying the properties of the functions that transform a member of the class into another.

A category is an algebraic structure consisting of a collection of *objects*, linked together by a collection of *arrows* (*morphisms*) that have two basic properties: the ability to compose the arrows associatively and the existence of an identity arrow for each object. *Objects* and *arrows* may be comprehended as abstract entities of any kind. The theory generalizes all of mathematics in terms of objects and arrows independent of what the objects and arrows represent.

In fact many branches of modern mathematics could be conveniently described in terms of categories; for example, category of sets, category of relations, category of groups, etc., and most importantly, doing so often reveals deep insights and similarities between seemingly different areas of mathematics (see [3,4,5,6,7,8,9] for details).

In view of the fact that a computer is not good at viewing concrete diagrams, category theory is being extensively used in computer science mainly because it offers a constructive mathematical structure to describe an object.

With these developments, category theory became an autonomous field of research (see [10] for the development of the theory of categories).

In addition, we invariably encounter with systems which contain objects with repeated elements or attributes (for example, groups of people, systems of elementary particles, etc., having two or more elements with the same property). We need a (formal) mathematical structure to model this kind of data. In the recent years such mathematical structures have been developed which are in general called *multiset-based structures*. Note that a *multiset* is a well defined collection of objects in which repetition of elements is considered significant. Accordingly, sets are merely special instances of multisets.

Applications of multiset abound, especially in mathematics and computer science, (see [11,12] for details). In this paper we attempt to present category of multisets.

2 Categories

A Category C consists of the following

(i) A Class $Ob(\boldsymbol{C})$ of *objects*.
(ii) A class $hom(\boldsymbol{C})$ of *morphisms or maps or arrows*.
(iii) A binary operation called *composition of morphisms*, such that for any three objects A, B and C, we have $hom(B, C)\ x\ hom(A, B) \rightarrow hom(A, C)$, (i.e., the composite of $f: A \rightarrow B$ and $g: B \rightarrow C$ is written $g\ o\ f: A \rightarrow C$ or $gf: A \rightarrow C$), governed by two axioms

(a) Associativity: if $f: A \rightarrow B, g: B \rightarrow C$ and $h: C \rightarrow D$ then $h\ o\ (g\ o\ f) = (h\ o\ g)\ o\ f$, and
(b) Identity: For every object X, there exists a morphism $1_X : X \rightarrow X$ called the *identity* morphism for X, such that for every morphism $f: A \rightarrow B$, we have $1_B\ o\ f = f = f\ o\ 1_A$.

The type of objects depends upon a chosen mathematical structure. For example, in set theory the objects are sets, in group theory the objects are groups and in graph theory the objects are graphs.

Morphisms are structure-preserving maps. It is an abstraction derived from structure preserving mappings between two mathematical structures. Morphism is an arrow linking an object called the *domain* to another object called the *codomain*. The notion of morphism can connotes differently depending on the type of space chosen. In set theory morphisms are functions, in group theory they are group homomorphisms and in graph theory they are graph homomorphisms.

2.1 Properties of Morphisms

A morphism $f: A \rightarrow B$ in a category $\boldsymbol{C}$ is called

(i) A *monomorphism* if it is left cancellable i.e., for every pair of morphisms $g, h: C \rightarrow A$ if $f\ o\ g = f\ o\ h \Rightarrow g = h$.
(ii) A *split monomorphism (or section or coretraction)* if it is left invertible i.e., there exists a morphism $g: B \rightarrow A$ such that $g\ o\ f = 1_A$. In that case A is called a *retract* of B.
(iii) An *epimorphism* if it is right cancellable i.e., for every pair of morphisms $g, h: B \rightarrow C$, $g\ o\ f = h\ o\ f \Rightarrow g = h$;
(iv) A *split epimorphism (or retraction)* if it is right invertible i.e., there exists a morphism $g: B \rightarrow A$ such that $f\ o\ g = 1_B$.
(v) A *bimorphism* if it is both a monomorphism and an epimorphism.
(vi) An *isomorphism* if it is both split monomorphism and split epimorphism i.e., if it is invertible (see [4] for details).

3 Multisets

Let us recall that a standard (or ordinary) set is a well defined collection of disjoint elements. In course of time, especially in order that set theoretic tools could be made applicable to solve problems which would require some generalizations of too restrictive conditions intrinsic to ordinary sets, various *nonstandard set theories* have appeared. In particular, if multiple but finite occurrences of any element of a set are allowed, we get a generalization of the notion of a set which is called a *Multiset (mset, for short)*.

As suggested in [13], collections admitting objects with finite multiplicities could be viewed in two different ways: Sets with distinguishable repeated elements (e.g., people or vehicles sharing a common property), called *multisets;* and sets with indistinguishable repeated elements (e.g., *soup* of elementary particles), called *multinumbers*. However, in the literature of multisets, Monro's terminology is usually reversed. Syropoulos [14], in particular, calls the former *Real multiset* and the latter *multiset*.

It is important to emphasize that both the concepts viz. real multisets and multisets are associated with a set equipped with an *equivalence relation* or a *function*, respectively. However, taking developments pertaining to admitting multiple occurrences of an object in a system, it is the notion of multiset which has mostly been exploited. In this paper, particularly to develop category of multisets, emphasis is on the notion of multisets associated with a set and an equivalence relation.

A Multiset is an unordered collection of objects in which, unlike a standard (Cantorian) set, duplicates or multiples of objects are admitted. In other words, an mset is a collection in which objects may appear more than once and each individual occurrence of an object is called its *element.* All duplicates of an object in an mset are indistinguishable. The objects of an mset are the distinguishable or distinct elements of the mset. The distinction made between the terms *object* and *element* does enrich the multiset language.

The use of square brackets to represent an mset is quasi-general. Thus, an mset containing one occurrence of a, two occurrences of b, and three occurrences of c is notationally written as $[[a,b,b,c,c,c]]$ or $[a,b,b,c,c,c]$ or $[a,b,c]_{1,2,3}$ or $[a,2b,3c]$ or $[a.1,b.2,c.3]$ or $[1/a,2/b,3/c]$ or $[a^1,b^2,c^3]$ or $[a^1\ b^2\ c^3]$. For convenience, the curly brackets are also used in place of the square brackets.

For various application purposes, we may regard a multiset $[a,a,b]$ as being really of the form $[a,a',b]$ where a and a' are different objects of the same sort and b is of different sort from that of a and a'. In this regard, when elements of multisets are considered, elements of distinct sorts will generally be denoted by distinct letters and elements of the same sort will be denoted by the same letter with dashes distinguishing different elements of that sort (see [13] for details).

Formally, a multiset A is a pair (A_0,ρ), where A_0 is a set and ρ an equivalence relation on A_0. The set A_0 is called *the field of the multiset*. Elements of A_0 in the same equivalence class will be said to be of the same sort, elements in different equivalence classes will be said to be of different sorts. For example, an mset $[a^2,b,c^3,d]$ is the multiset $[a,a',b,c,c',c'',d]$, where all are seen to behave as different objects; but a,a' are of the same sort and c,c',c'' are also of the same sort but different from that of a,a', while b,d are each of different sorts from the others. In other words, various equivalence classes determine the sorts. Whenever a multiset A is mentioned, its field will usually be denoted A_0. The pair (A_0,ϕ), where A_0 is a set and ϕ denote the empty relation on A_0, is actually an ordinary set.

3.1 Multiset Functions

Let $A = (A_0,\rho)$ and $B = (B_0,\sigma)$ be multisets. A multiset function (or morphism) from A to B, written as $f: A \to B$, is a function $f: A_0 \to B_0$ which respects sorts i.e., if $a,a' \in A_0$ and $a\rho a'$, then $f(a)\sigma f(a')$. For example, $f: [a,b,c] \to [d,d',e]$ defined by $f(a) = d', f(b) = e, f(c) = d$; $g: [a,a',b] \to [c,c',d,e,f]$ defined by $g(a) = c, g(b) = d, g(a') = c'$; and $g(a) = g(a') = d, g(b) = c$ are all multiset functions, while $h: [a,a',b] \to [c,c',d]$ defined by $h(a) = c', h(a') = d$ and $h(b) = c$ is not an mset function since $a,a' \in A_0$ and $a\rho a'$ but is not the case $h(a)\sigma h(a')$.

Let $\theta = (\phi_0,\phi)$, where ϕ_0 is an empty set and ϕ an empty relation on ϕ_0. Then, θ is said to be an *empty* multiset. That is, an empty set together with an empty equivalence relation defined on it determine an empty multiset.

4 Category of Multisets

Multisets (considered as objects) and multiset functions (considered as morphisms) together determine the category of multisets, denoted **Mul**. That is, in **Mul**, the *objects* are *multisets* and the *morphisms* are *multiset functions*.

Typically, multisets can themselves be regarded as categories such that if A is a multiset and $a, b \in A$, there is an arrow from a to b if a and b are of the same sort, and no arrows from a to b if a and b are of different sorts.

Theorem 4.1: In **Mul**, the monomorphisms are exactly the injective functions.

Proof: Suppose $f: A \to B$ is injective and $f \, o \, g = f \, o \, h$ for $g, h: X \to A$. Then for every $x \in X, (f \, o \, g)(x) = (f \, o \, h)(x) \Rightarrow f(g(x)) = f(h(x)) \Rightarrow g(x) = h(x)$ since f is injective. Thus $g = h$ i.e., f is a monomorphism.

Theorem 4.2: If $f: A \to B$ and $k: B \to D$ are injective, then $k \, o \, f$ is also injective. That is, the composite of two injective arrows is also injective.

Proof: let $a_1, a_2 \in A$ such that $(k \, o \, f)(a_1) = (k \, o \, f)(a_2)$ and hence $k(f(a_1)) = k(f(a_2))$. Now, since k is injective, $f(a_1) = f(a_2)$ and since f is injective, $a_1 = a_2. \Rightarrow k \, o \, f$ is injective.

Theorem 4.3: In **Mul**, a split monomorphism is a monomorphism.

Proof: Let $f: A \to B$ be a split monomorphism (i.e., $\exists \, k: B \to A$ such that $k \, o \, f = 1_A$) and suppose $f \, o \, g = f \, o \, h$ for $g, h: C \to A$. We are to show that $g = h$. Now for each $c \in C, g(c) = (1_A \, o \, g)(c) = \big((k \, o \, f) o \, g\big)(c) = \big(k \, o \, (f \, o \, g)\big)(c) = \big(k \, o \, (f \, o \, h)\big)(c) = \big((k \, o \, f) o \, h\big)(c) = (1_A \, o \, h)(c) = h(c) \Rightarrow g = h.$ Since $g = h, f$ is a monomorphism.

Theorem 4.4: The composite of two monomorphisms in **Mul** is a monomorphism.

Proof: Let $f: A \to B$ and $p: B \to D$ be monomorphisms. We are to show that if $(p \, o \, f) \, o \, g = (p \, o \, f) \, o \, h$ for $g, h: C \to A$, then $g = h$ i.e., $p \, o \, f$ is a monomorphism.

Now for each $c \in C, \big((p \, o \, f) o \, g\big)(c) = \big((p \, o \, f) o \, h\big)(c) \Rightarrow \big((p \, o \, f) g\big)(c) = \big((p \, o \, f) h\big)(c) \Rightarrow p\Big(f\big(g(c)\big)\Big) = p\Big(f\big(h(c)\big)\Big).$ Since p and f are monomorphisms, we have $g(c) = h(c)$ i.e., $g = h$. Hence $p \, o \, f$ is a monomorphism.

Theorem 4.5: In **Mul**, a surjective arrow is an epimorphism.

Proof: Let $f: A \to B$ be surjective and $g \, o \, f = h \, o \, f$ for $g, h: B \to C$. Now, in order to show that f is an epimorphism, we need to show that $g = h$. As for every $a \in A, (g \, o \, f)(a) = (f \, o \, h)(a)$ i.e., $g(f(a)) = h(f(a))$, we have $g = h$.

Theorem 4.6: In **Mul,** a split epimorphism is an epimorphism.

Proof: Let $f: A \to B$ be a split epimorphism i.e., $\exists \, k: B \to A$ such that $f \, o \, k = 1_B$. Let $g \, o \, f = h \, o \, f$ for $g, h: B \to X$. We are to show that $g = h$.

Now, for each $b \in B$, we have

$$g(b) = (g \, o \, 1_B)(b) = \big(g \, o \, (f \, o \, k)\big)(b) = \big((g \, o \, f) o \, k\big)(b) = \big((h \, o \, f) o \, k\big)(b)$$
$$= \big(h \, o \, (f \, o \, k)\big)(b) = (h \, o \, 1_B)(b) = h(b).$$

Hence, $g = h$ i.e., f is an epimorphism.

Theorem 4.7: In **Mul**, the composite of two epimorphisms is an epimorphism.

Proof: Let $f: A \to B, q: B \to C$ be epimorphisms and $g\ o\ (q\ o\ f) = h\ o\ (q\ o\ f)$. We need to show that $g = h$ for every pair $g, h: C \to D$. Now, for each $a \in A$, we have $(g\ o\ (q\ o\ f))(a) = (h\ o\ (q\ o\ f))(a)$ which implies $g(q(f(a))) = h(q(f(a)))$, and since f and q are epimorphisms; $g = h$.

Theorem 4.8: In **Mul**, the composite of two bimorphisms is a bimorphism.

Proof: Let $f: A \to B, k: B \to D$ be bimorphisms. We have, $f\ o\ g = f\ o\ h \Rightarrow g = h$ for $g, h: C \to A$; $g\ o\ f = h\ o\ f \Rightarrow g = h$ for $g, h: B \to X$; $k\ o\ g = k\ o\ h \Rightarrow g = h$ for $g, h: C \to B$; and $g\ o\ k = h\ o\ k \Rightarrow g = h$ for $g, h: D \to X$. We need to show that $(k\ o\ f)\ o\ g = (k\ o\ f)\ o\ h \Rightarrow g = h$ for $g, h: C \to A$, and $g\ o\ (k\ o\ f) = h\ o\ (k\ o\ f) \Rightarrow g = h$ for $g, h: D \to X$.

Now, for each $c \in C, ((k\ o\ f)\ o\ g)(c) = (k\ o\ (f\ o\ g))(c) = (k\ o\ (f\ o\ h))(c)\ i.e., k(f(g(c))) = k(f(h(c)))$. Since k and f are bimorphisms, we have $g(c) = h(c)$ i.e., $g = h$.

Moreover, for each $a \in A, (g\ o\ (k\ o\ f))(a) = ((g\ o\ k)\ o\ f)(a) = ((h\ o\ k)\ o\ f)(a)$ i.e., $g(k(f(a))) = h(k(f(a)))$. Since f and k are bimorphisms, we have $g = h$. Hence, $k\ o\ f$ is a bimorphism.

Theorem 4.9: In **Mul, every** isomorphism is a bimorphism.

Proof: Let $f: A \to B$ be an isomorphism i.e., $\exists\ k: B \to A$ such that $k\ o\ f = 1_A, f\ o\ k = 1_B$ and suppose $f\ o\ g = f\ o\ h$ for $g, h: C \to A$; $g\ o\ f = h\ o\ f$ for $g, h: B \to C$: We need to show that $g = h$ in both the cases.

Now, for each
$a \in A, g(a) = (1_A\ o\ g)(a) = ((k\ o\ f)\ o\ g)(a) = (k\ o\ (f\ o\ g))(a) = (k\ o\ (f\ o\ h))(a) = ((k\ o\ f)\ o\ h)(a) = (1_A\ o\ h)(a) = h(a)$ i.e., $g = h$.

Also, for each
$b \in B, g(b) = (g\ o\ 1_B)(b) = (g\ o\ (f\ o\ k))(b) = ((g\ o\ f)\ o\ k)(b) = ((h\ o\ f)\ o\ k)(b) = (h\ o\ (f\ o\ k))(b) = (h\ o\ 1_B)(b) = h(b)$ i.e., $g = h$. Hence f is a bimorphism.

Moreover, the converse of this theorem need not hold in **Mul**.

Definition 4.1 Dual of a Mul

The dual of a **Mul**, denoted **Mul**^{op}, is a category in which

(i) Objects and arrows are those of **Mul**;
(ii) If $f: A \to B$ is in **Mul**, then $f: B \to A$ is in **Mul**^{op} i.e., the source and target are reversed;
(iii) If $h = gof$ in **Mul**, then $h = fog$ in **Mul**^{op}; and
(iv) The identity arrows in **Mul** and **Mul**^{op} are the same.

Definition 4.2 Objects in Mul

An object I in a category $\boldsymbol{C}$ is said to be an *initial* object of $\boldsymbol{C}$ if for every object X of $\boldsymbol{C}$ there is a unique morphism $I \to X$. In this definition, the nature of a particular object I is described in terms of its relation to all other objects X in the category. Since for every object X in **Mul** there is a unique morphism $\theta \to X$, the empty multiset θ can be regarded as an initial object in **Mul**.

Also, an object T in a category $\boldsymbol{C}$ is said to be a *terminal* object of $\boldsymbol{C}$ if for each object X of $\boldsymbol{C}$ there is exactly one morphism $X \to T$. In **Mul**, an object satisfying this universal mapping property is an one-element multiset, denoted $\mathbf{1}$. Indeed, for each object X of **Mul**, there is a unique morphism $X \to \mathbf{1}$. Hence, any singleton mset is a terminal object in **Mul**. As we may have more than one singleton, they must be isomorphic.

Since a zero object is an object that is both initial and terminal, there is no zero object in **Mul**.

Thus the category of multisets has a unique initial object, but many terminal objects which are isomorphic to each other. The proof follows:

Theorem 4.10: All singletons in **Mul** are isomorphic.

Proof: Suppose M and M' are singletons. Thus, there are morphisms $g: M \to M'$ and $h: M' \to M$. The composite $g \circ h$ is an arrow $M' \to M'$ which is the identity $1_{M'}$. Since M' is a singleton, there is only one arrow $M' \to M'$, and hence, $g \circ h = 1_{M'}$
Similarly, $h \circ g = 1_M$.

Definition 4.3 Points of an Object in Mul

A point of an object X in a category $\boldsymbol{C}$ is an arrow $\mathbf{1} \to X$, where $\mathbf{1}$ is terminal. Hence, in the category **Mul**, the points of an object A are precisely the elements of A. Suppose $A = [a, a', b]$ is an object in the category **Mul**, then A has three points and, each point of A can be seen to *point to* exactly one element of A and every element of A is the value of exactly one point of A. Moreover, if $f: A \to B$ and a is a point of A, then fa is a point of B. The terminal object in **Mul** has one point. The terminal object $\mathbf{1}$ in **Mul** helps in separating arbitrary arrows. For the parallel arrows $f, g: A \to B$, if for each point a of A we have $fa = ga$, then $f = g$. Thus, if two arrows *agree* on points, they are the same arrows. For example, given $f: A \to B$ and $g: A \to B$, if $fa = ga$ for every point $a: \mathbf{1} \to A$, then we can conclude that $f = g$. In other words, if $f \neq g$, then there exists at least one point $a: \mathbf{1} \to A$ for which $fa \neq ga$. Similar to Set, two arrows in **Mul** are equal if they have the same domain and codomain and if they have the same value at every element of their domain.

Definition 4.4 Product of Objects in Mul

An object $A\text{x}B$, together with a pair of maps $A\text{x}B \xrightarrow{p1} A, A\text{x}B \xrightarrow{p2} B$, is called the *product* of the objects A and B if for every object X and each pair of maps $X \xrightarrow{f1} A, X \xrightarrow{f2} B$, there is a unique map $X \xrightarrow{f} A\text{x}B$ such that $f_1 = p_1 f, f_2 = p_2 f$. Diagramatically, it can be put as shown in Fig. 1 below.

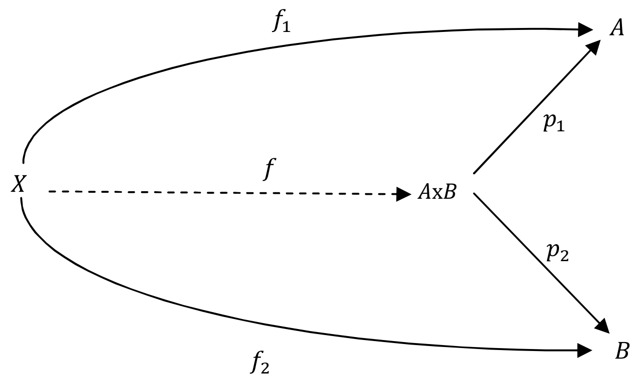


Fig. 1. Product of two objects

Since f is uniquely determined by f_1 and f_2, we can denote it by $\langle f_1, f_2 \rangle$. The maps p_1 and p_2 are called *projection maps* for the product or simply *projections*, (i.e., projections of the product to its factors).

In **Mul**, each point of the product of the objects A, B can be uniquely represented in the form $\langle a, b \rangle$, where a is a point of A and b is a point of B. That is, the points of the product of two objects are the pairs of points, one from each factor. We recall that if two arrows agree on points, they are the same i.e., an arrow is completely determined by the values of its points. Therefore, as soon as we know the points of the objects, we can determine their product.

Theorem 4.11: In **Mul**, if an object P and the functions $k_1: P \rightarrow A$ and $k_2: P \rightarrow B$ have the product property, then P is isomorphic to AxB and, there is exactly one isomorphism $f: P \rightarrow A\text{x}B$ with $k_1 = p_1 \, o \, f$ and $k_2 = p_2 \, o \, f$. Diagramatically, it can be put as shown in Fig. 2 below.

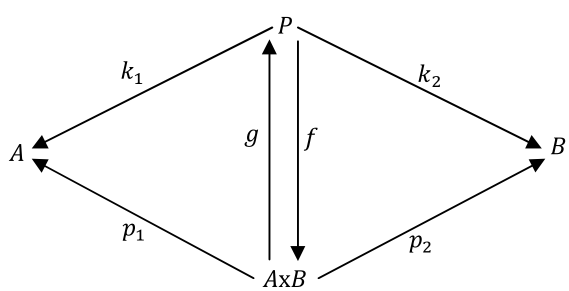


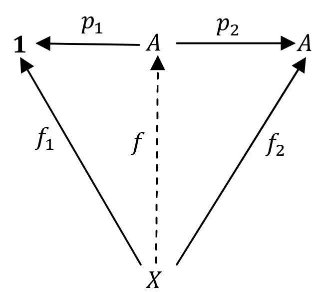
Fig. 2. Isomorphism between two objects having product properties

Proof: Using the product property of P, k_1 and k_2, we define $g: A\text{x}B \rightarrow P$. Let $f: P \rightarrow A\text{x}B$ with $k_1 = p_1 \, o \, f$ and $k_2 = p_2 \, o \, f$, and $g: A\text{x}B \rightarrow P$ with $p_1 = k_1 \, o \, g$ and $p_2 = k_2 \, o \, g$. For a unique isomorphism between them, we need to show that $g \, o \, f = 1_P$ and $f \, o \, g = 1_{A\text{x}B}$ hold. We have, $k_1 \, o \, (g \, o \, f) = (k_1 \, o \, g) \, o \, f = p_1 \, o \, f = k_1$ and, $k_2 \, o \, (g \, o \, f) = (k_2 \, o \, g) \, o \, f = p_2 \, o \, f = k_2$. Since, $1_P: P \rightarrow P$ is the only arrow that will compose with k_1 and k_2 to give k_1 and k_2 respectively $g \, o \, f$ must be the identity arrow i.e., $g \, o \, f = 1_P$. Similarly, since $p_1 \, o \, (f \, o \, g) = (p_1 \, o \, f) \, o \, g = k_1 \, o \, g = p_1$ and, $p_2 \, o \, (f \, o \, g) = (p_2 \, o \, f) \, o \, g = k_2 \, o \, g = p_2$; we have $f \, o \, g = 1_{AxB}$.

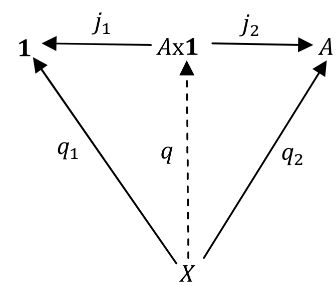
Theorem 4.12: In **Mul**, every object A is isomorphic to the product Ax**1**, where **1** is terminal.

Proof: An object A, with $p_1: A \rightarrow \mathbf{1}$ and $p_2: A \rightarrow A$ as projections, has the product property, i.e., A with $p_1: A \rightarrow \mathbf{1}$ and $p_2: A \rightarrow A$ as projections such that $f_1 = p_1 \, o \, f, \; f_2 = p_2 \, o \, f$ as in Fig. 3(a) below.

Clearly $A\mathrm{x}\mathbf{1}$, with $j_1: A\mathrm{x}\mathbf{1} \rightarrow \mathbf{1}$ and $j_2: A\mathrm{x}\mathbf{1} \rightarrow A$ as projections, has the product property i.e., an mset $A\mathrm{x}\mathbf{1}$ with $j_1: A\mathrm{x}\mathbf{1} \rightarrow \mathbf{1}$ and $j_2: Ax\mathbf{1} \rightarrow A$ as projections such that $q_1 = j_1 \, o \, q, \; q_2 = j_2 \, o \, q$ as in Fig. 3(b) below.



a



b

Fig. 3. Objects having product properties

Taking the aforesaid facts and Fig. 4 below into consideration, it follows from theorem 4.11 A is isomorphic to $A\mathrm{x}\mathbf{1}$, where $\mathbf{1}$ is terminal in **Mul**.

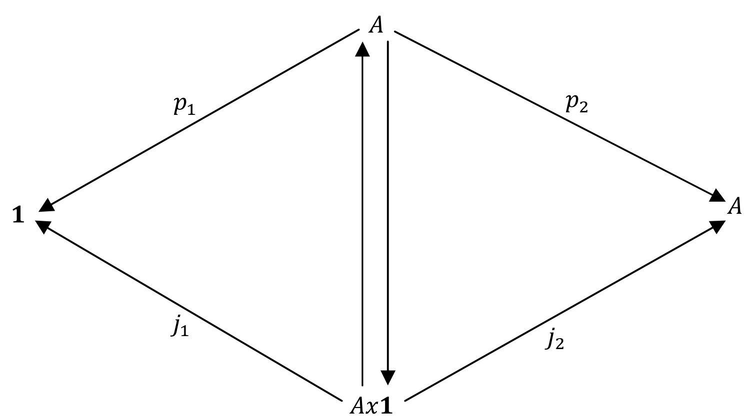


Fig. 4. Isomorphism between an object and its product with a terminal object

4.1 Dual of Product-Projections

Dualizing the notion of product-projections, we get:

An object $A + B$ together with a pair of maps $A \overset{i1}{\rightarrow} A + B, B \overset{i2}{\rightarrow} A + B$ in a category is said to be a coproduct of A and B if for each pair of maps $A \overset{g1}{\rightarrow} Y, B \overset{g2}{\rightarrow} Y$, there is a unique map $A + B \overset{g}{\rightarrow} Y$ such that $g_1 = g \, o \, i_1$ and $g_2 = g \, o \, i_2$.

That is, the following in Fig. 5 commutes.

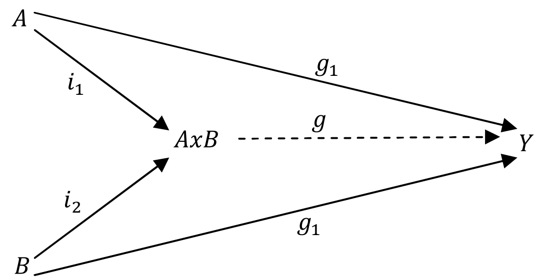


Fig. 5. Coproduct of two objects

Thus, $g_1 = g\ o\ i_1$ and $g_2 = g\ o\ i_2$, where i_1 and i_2 are called *injections* for the coproduct. Recall that in **Mul**, the coproduct is called the *sum*. The functorial extension of this operation from objects to all functions follows just as in the case of the product operation. Also, *Sums* have the property that any point of $A + B$ comes via injection from a point of exactly one of A, B.

Theorem 4.13: There is exactly one isomorphism with injections between any two *Sums* of the objects A and B in **Mul**.

Proof: Suppose Q, with $j_1: A \to Q$ and $j_2: B \to Q$ as injections, has the sum property like that of $A + B$, i_1 and i_2, then there is an isomorphism $k: A + B \to Q$ such that $j_1 = k\ o\ i_1$ and $j_2 = k\ o\ i_2$. Using the Sum property we define $h: Q \to A + B$ with $i_1 = h\ o\ j_1$ and $i_2 = h\ o\ j_2$. Diagramatically, it can be put as in Fig. 6 below.

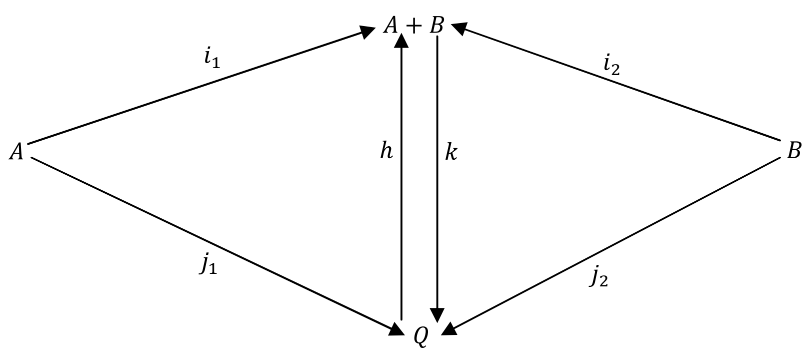


Fig. 6. Isomorphism between two sums of objects A and B

We need to show that $k\ o\ h = 1_Q$ and, $h\ o\ k = 1_{A+B}$.

Since, $(k\ o\ h) o\ j_1 = k\ o\ (h\ o\ j_1) = k\ o\ i_1 = j_1$and $(k\ o\ h)\ o\ j_2 = k\ o\ (h\ o\ j_2) = k\ o\ i_2 = j_2$; and also $1_Q\ o\ j_1 = j_1$ and $1_Q\ o\ j_2 = j_2$; by the uniqueness property, we have $k\ o\ h = 1_Q$. Similarly, $h\ o\ k = 1_{A+B}$.

5 Conclusions

It is shown that Multisets as objects and multiset functions as morphisms determine a category denoted **Mul**. In **Mul**, the monomorphisms are exactly the injective functions while the epimorphisms are the surjective functions and every split monomorphism is a monomorphism and every split epimorphism is an epimorphism. It is also proved that the composite of two monomorphisms, epimorphisms and biomorphisms are monomorphisms, epimorphisms and biomorphisms, respectively. It is further illustrated that similar to **Set** if two arrows *agree* on points, they are the same arrows.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Pombo O, Symons J, Torres JM. Neurath and the unity of science. An Introduction, Logic, Epistemology, and the Unity of Science. 2011;18:1-11.

[2] Marquis JP. Category theory, Metaphysics Research Lab, CSLI (Centre for the study of Language and Information). Stanford University. 2010;1-22.

[3] Barr M, Wells C. Category theory. Lecture Notes for ESSLLI (European Summer School of Logic, Language and Information). 1999;1-128.

[4] Laan V. Introduction to category theory. Lecture Notes. 2003;1 -52.

[5] Fokkinga MM. A gentle introduction to category theory: The calculational approach. University of Twente, the Netherlands. 1994;1-58.

[6] Landry E. Category theory: The language of mathematics, proceedings of the 1998 biennial meetings of the philosophy of science association. University of Chicago Press. 1999;66:S14-S27.

[7] Lawvere FW. An elementary theory of the category of sets (Long version) with commentary. Reprints in Theory and Applications of Categories. 2005;11:1–35.

[8] Cockett JRB. Categories and computability. University of Calgary, Canada. 2010;1-93.

[9] Oosten JV. Basic category theory. Lecture Notes, Utrecht university, Netherlands, Revised Edition. 2002;1- 86.

[10] Singh D, Isah AI, Ibrahim AM. A survey of the development of the theory of categories. International Journal of Innovation Management and Technology. 2012;3(2):156-159.

[11] Singh D, Ibrahim AM, Tella Y, Singh JN. An overview of the applications of multisets. NOVI Sad J. Math. 2007;37(2):73–92.

[12] Blizard WD. The development of multiset theory. Modern Logic. 1991;1:319–352.

[13] Monro GP. The concept of multiset. Zeitschrift für mathematische Logik und Grundlagen der mathematic [mathematical logic quarterly]. 1987;33(2):171-178.

[14] Syropoulos A. Mathematics of multisets. Calude CS, et al. (Eds.): Multiset Processing, LNCS 2235. 2001;347-358.

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