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A Parabolic Transform and Some Stochastic Ill-Posed Problems

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Abstract

It is well known that the Cauchy problem for elliptic partial differential equations is ill-posed. The question, which arises, how a priori knowledge about solutions can bring about stability? A parabolic transform is defined to discuss the stability of the Cauchy problem for some stochastic partial differential equations under a priori knowledge about solutions. With the help of the parabolic transform, existence results are established for general linear and nonlinear stochastic partial differential equations, without any restrictions on the characteristic forms. Many physical and engineering problems in areas like seismology, geophysics and biology require the solutions of ill-posed problems. The Cauchy problem for general stochastic differential equations has many different important applications with amazing range.

Keywords: Parabolic transform; cauchy problem for general stochastic partial differential equations; stability of solutions.

AMS Subject Classifications: 34K30, 26 A 33, 60.H15.

1 Introduction

Let $W(t)$ be a standard Brownian motion defined on the filtered probability space $(\Omega, F, F_t P)$, where F is a σ-algebra of subsets of Ω, $(F_t : 0 \leq t \leq T)$ is a right σ-continuous, increasing family

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of subsets of sub σ- algebra of F and P is a probability measure defined on F.

Consider the stochastic partial differential equation.

$$du(x,t) = L_1(t)u(x,t)dt + L_2(t)u(x,t)dW(t), \tag{1.1}$$

with the initial condition

$$u(x,0) = \phi(x), \tag{1.2}$$

where

$$L_1(T) = \sum_{|q|\le m} a_q(t)D^q, L_2 = \sum_{|q|\le m} b_q(t)D^q,$$

$$D^q = D_1^{q_1} D_2^{q_2} \cdots D_k^{q_k}, D_i f(x) = \frac{\partial}{\partial x_i} f(x),$$

$$D_i f(y) = \frac{\partial}{\partial y_i} f(y), q = q_1 \cdots q_k,$$

q is a multi-ind $|q| = q_1 + \cdots + q_k$, $-\infty < x_i < \infty$, $i = 1, 2, \cdots k$, $0 \le t \le T$, $\{a_q(t), b_q(t) : |q| \le m\}$ are families of continuous functions on $[0, T]$ and ϕ is an F_0 stochastic process such that $\phi \in H$, almost surely, a.s., H is the set of all square integrable functions on the k-dimensional Euclidean space R^k. In section 2, we shall study the correct formulation of the stochastic Cauchy problem (1.1), (1.2), under priori knowledge about solutions, without any restrictions on the characteristic forms of the operators $L_1(t)$ and $L_2(t)$, (see [1-3]), In section 3, we shall study under suitable conditions, the existence and uniqueness of the solutions of the Cauchy problem (1.1), (1.2) and for more general nonlinear stochastic partial differential equations of the form:

$$du = f(x,t,Bu)dt + g(x,t,Bu)dW(t), \tag{1.3}$$

with the initial condition

$$u(x,0) = \phi(x), \tag{1.4}$$

where $Bu = (Bu_1, ..., Bu_r)$, $B_1u, ..., B_ru$ are some of the partial derivatives $D^q u$, for $|q| \le m$.

It is supposed that: $f, g : R^k X[0,T]XR^r \to R$ are continuous and satisfy the following conditions:

$$\|f(x,t,Bu) - f(x,t,Bu)\| \le K \sum_{|q|\le m} \|D^q u - D^q v\|, \tag{1.5}$$

$$\|g(x,t,Bu) - g(x,t,Bu)\| \le K \sum_{|q|\le m} \|D^q u - D^q v\|, \tag{1.6}$$

$$\|f(x,t,Bu)\| \le K\left[1 + \sum_{|q|\le m} \|D^q u\|\right] \qquad |g(x,t,Bu)\| \le K\left[1 + \sum_{|q|\le m} \|D^q u\|\right], \tag{1.7}$$

for all x in $R^k, 0 \le t \le T$, for some constant $K > 0$, $\|u(\cdot)\|^2 = \int u^2(x)dx$, where the integrals with respect to x are taken over R^k. It is supposed that the derivatives $D^q u$, $|q| \le m$ are in the generalized sense of Sobolev.

2 Parabolic Transform and Stability

Consider the following Cauchy problem:

$$\frac{\partial \psi(x,t)}{\partial t} = L\psi(x,t), \tag{2.1}$$

$$\psi(x,0) = \psi_0(x), \tag{2.2}$$

where $L = [D_1^2 + D_2^2 + \cdots + D_k^2]^{2N+1}$, N is a sufficiently large positive integer, $\psi_0 \in H$. The solution of the Cauchy problem (2.1), (2.2) is given by:

$$\psi(x,t) = \int G(x-y,t)\psi_0(y)dy,$$

where G is the fundamental solution of the Cauchy problem (2.1), (2.2). For sufficiently large N, We can find a constant $K > 0$ and $0 < \gamma < \frac{1}{2}$ such that

$$\|D^q\psi(\cdot,t)\| \leq Kt^{-\gamma}\|\psi(\cdot,t)\|, \tag{2.3}$$

for all $|q| \leq m$. It is clear that

$$\int G(x,t)dx = 1, \tag{2.4}$$

Let us define a parabolic transform v of a function u by

$$v(x,t) = \int G(x-y,ct)u(y,t)dy, \tag{2.5}$$

where $0 < c < 1$.

By a solution of the stochastic Cauchy problem (1.1), (1.2), we mean an F_t-adapted stochastic process u, which have the following properties:

(i) $\|u(\cdot,t)\|$ is continuous on $[0,T]$, almost surely,

(ii) $D^q u$ exists for every q, $|q| \leq m$, in the sense of Sobolev, $D^q u \in H$,

(iii) $E\left[\|D^q u(\cdot,t)\|^2\right] < \infty$, for all $|q| \leq m$, $t \in [0,T]$, where $E(X) = \int_\Omega X(w)dP$, is the expectation of a random variable X.

(iv) u satisfies the following equation:

$$u(x,t) = \phi(x) + \int_0^t L_1(s)u(x,s)ds + \int_0^t L_2(s)u(x,s)dW(s). \tag{2.6}$$

In the same manner, we define the solution of the stochastic Cauchy problem (1.3), (1.4).

It is supposed that, for every $x \in R^k$, the initial stochastic process ϕ is independent of the future of Brownian motion $W(t)$ beyond time $t = 0$, it is supposed also that $E\lfloor\|\phi\|^2\rfloor < \infty$.

We shall find an estimation of $E\lfloor\|u(\cdot,t)\|^2\rfloor$.

Theorem 1. Let u be a solution of the Cauchy problem (1.1), (1.2). Suppose that

$$E\lfloor\|\phi\|^2\rfloor < \epsilon^2, \tag{2.7}$$

where $\epsilon > 0$ is a sufficiently small number. If u satisfies the condition

$$E\lfloor\|Lu(\cdot,t)\|^2\rfloor \leq M, \tag{2.8}$$

where $M > 0$ is a constant independent of ϵ, then

$$E\lfloor\|u(\cdot,t)\|^2\rfloor \leq \frac{K}{(\ln\frac{1}{\epsilon})^\alpha}, \tag{2.9}$$

for some constants $K > 0$, $\alpha > 0$. The constants K and α are independent of ϵ.

Proof. Let v be the parabolic transform of u, defined by (2.5). Then

$$\begin{aligned} v(x,t) &= \int G(x-y),ct)\phi(y)dy + \int\int_0^t G(x-y,ct-cs)L_1(s)v(y,s)dyds \\ &+ \int\int_0^t G(x-y,ct-cs)L_2(s)v(y,s)dydW(s). \end{aligned} \tag{2.10}$$

Let $\widetilde{h}$ denote the Fourier transform of a function h defined by

$$\widetilde{h} = \left(\frac{1}{2\pi}\right)^{\frac{k}{2}} \int h(x)e^{-ix\cdot\sigma}dx, \qquad x\cdot\sigma = x_1\sigma_1 + \cdots + x_k\sigma_k$$

It is clear that:

$$\begin{aligned} \widetilde{v}_1(\sigma,t) &= \int_0^t \rho_1(\sigma,s)\widetilde{G}(\sigma,ct-cs)\widetilde{v}(\sigma,t)ds, \\ \widetilde{v}_2(\sigma,t) &= \int_0^t \rho_1(\sigma,s)\widetilde{G}(\sigma,ct-cs)\widetilde{v}(\sigma,t)dW(s), \end{aligned}$$

where $\widetilde{v}_1$ is the Fourier transform of v_1:

$$v_1(x,t) = \int_0^t \int L_1(s)G(x-y,ct-cs)v(y,s)dyds,$$

and $\widetilde{v}_2$ is the Fourier transform of v_2:

$$v_2(x,t) = \int_0^t \int L_2(s)G(x-y,ct-cs)v(y,s)dydW(s),$$

$$\widetilde{G}(\sigma,ct) = e^{-ct\rho(\sigma)},$$

$$\rho(\sigma) = [\sigma_1^2 + \cdots + \sigma_k^2]^{2N+1}, \rho_1(\sigma) = (2\pi)^{\frac{k}{2}} \sum_{|q|\le m} a_q(t)(i\sigma)^q, \rho_2(\sigma) = (2\pi)^{\frac{k}{2}} \sum_{|q|\le m} b_q(t)(i\sigma)^q.$$

It is easy to see that

$$\|\widetilde{v}_1(\cdot,t)\|^2 \le \frac{K}{c^\gamma}\int_0^t \frac{\|\widetilde{v}_1(\cdot,s)\|^2}{(t-s)^\gamma}ds. \tag{2.11}$$

It is clear that:

$$\begin{aligned} E\left[\|\widetilde{v}_2(\cdot,t)\|^2\right] &= \int E\left[\int_0^t \rho_2(\sigma,s)\widetilde{G}(\sigma,ct-cs)\widetilde{v}(\sigma,s)dW(s)\right]^2 d\sigma \\ &= \int\int_0^t E\left[\rho_2^2(\sigma,s)\widetilde{G}^2(\sigma,ct-cs)\widetilde{v}^2(\sigma,s)\right] dsd\sigma. \end{aligned}$$

Interchanging the order of integration and using Parseval's identity, we get

$$E\left[\|v_2(\cdot,t)\|^2\right] \le \frac{K}{c^\gamma}\int_0^t \frac{E\left\lfloor\|v(\cdot,s)\|^2\right\rfloor}{(t-s)^\gamma}ds \tag{2.12}$$

From (2.10), (2.11) and (2.12), one gets,

$$E\left[\|v(\cdot,t)\|^2\right] \le K\epsilon^2 + \frac{K}{c^\gamma}\int_0^t \frac{E\left\lfloor\|v(\cdot,s)\|^2\right\rfloor}{(t-s)^\gamma}ds \tag{2.13}$$

for some constant $K > 0$, $0 < \gamma < \frac{1}{2}$. Inequality (2.13) leads to

$$E\left[\|v(\cdot,t)\|^2\right] \le \frac{K\epsilon^2}{c^{2\gamma}} + \exp\left[\frac{K\epsilon^2}{c^{2\gamma}}\right].$$

Choose $c^{2\gamma} = \frac{2KT}{\ln \frac{1}{\epsilon}}$, we get

$$E\left[\|v(\cdot,t)\|^2\right] \le \epsilon^{1-\frac{t}{2T}}, \tag{2.14}$$

for all $0 \le t \le T$. Note that, $u(x,t) = v(x,t) - t\int_0^c \int G(x-y,\theta t)Lu(y,t)dyd\theta$. The last identity and (2.14) lead to the required result, [4-6]. □

3 Existence and Uniqueness of Solutions

In general the Cauchy problem (1.1),(1,2) or (1.3),(1,4) have no solutions, even in the deterministic case. But we shall find a dense set S in H such that if $\phi(x)$ in S, then those Cauchy problems (1.1), (1.2) and (1.3), (1.4) can be solved under suitable modifications of the nonlinear functions f and g. Suppose that:

$$D^q\phi \cdot \epsilon H, |q| \le m,$$

almost surely. Let us solve first equation (2.10) by using the method of successive approximations. To do this, we set

$$\begin{aligned} v_{n+1}(x,t) &= \int G(x-y,ct)\phi(y)dy + \int\int_0^t L_1(s)G(x-y,ct-cs)v_n(y,s)dyds \\ &+ \int\int_0^t L_2(s)G(x-y,ct-cs)v_n(y,s)dydW(s). \end{aligned}$$

The zero approximation $v_0(x,t)$ is taben to be zero. Thus$E\left[\|\widetilde{v}_1(\cdot,t)\|^2\right] \le E\left[\|\phi\|^2\right] \le M$, for some constat $M > 0$.

Using similar steps as in theorem (2.1) and Martingale inequality, one gets

$$\begin{aligned} E\left[\max_{0\le t\le T} \|v_n^*(\cdot,t) - v_{n-1}^*(\cdot,t)\|^2\right] &\le \frac{4K}{c^\gamma}\int_0^T \frac{\|v_n(\cdot,s) - v_{n-1}(\cdot,s)\|^2}{(T-s)^\gamma}ds \\ &\le 4Z_n(T). \end{aligned}$$

where

$$v_{n+1}^* = \int\int_0^t L_2(s)G(x-y,ct-cs)v_n(y,s)dydW(s),$$

$$Z_n(t) = M\left[\frac{K(1-\gamma)}{c^\gamma}\right]^n \frac{t^n}{\Gamma(n(1-\gamma)+1)}.$$

Now, since

$$\begin{aligned} P\left[\max_{0\le t\le T} \|v_{n+1}(\cdot,t) - v_n(\cdot,t)\|^2 > \frac{1}{2^n}\right] &\le 2^n E[\max_{0\le t\le T} \|v_{n+1}(\cdot,t) - v_n(\cdot,t)\|^2] \\ &\le 2^n Z_n(T), \end{aligned}$$

and

$$\sum_{n=1}^{\infty} 2^n Z_n(T) < \infty,$$

it follows that $P[\max_{0\le t\le T} \|v_{n+1}(\cdot,t) - v_n(\cdot,t)\| > \frac{1}{2^n}] = 0$, infinitely often. Thus the series $\sum_{j=0}^n \|v_{j+1}(\cdot,t) - v_j(\cdot,t)\|$, uniformly converges on $[0,T]$.

So the sequence $\{v_n\}$ converges in H, uniformly on $[0.T]$, to a stochastic process v, which satisfies equation (2.10). We have, formally,

$$\begin{aligned} D^q v(x,t) &= \int G(x-y,ct)D^q\phi(y)dy \\ &+ \int_0^t L_1(s)G(x-y,ct-cs)D^q v(y,s)dyds \\ &+ \int_0^t L_2(s)G(x-y,ct-cs)D^q v(y,s)dydW(s). \end{aligned}$$

Thus the generalized derivatives $D^q v$ exist for all $|q| \leq m$. It is clear also that

$$E[\|D^q v\|^2] \leq \frac{K}{c^{2\gamma}} E[\|\phi\|^2] \exp \frac{Kt}{c^{2\gamma}},$$

for all $|q| \leq m$, $t \in [0,T]$.

To prove the uniqueness, let us suppose that v and v^* satisfies equation (2.10). We have

$$E[\|v(\cdot,x) - v^*(\cdot,t)\|^2] \leq K \int_0^t \frac{E[\|v(\cdot,s) - v^*(\cdot,s)\|^2]}{c^\gamma (t-s)^\gamma},$$

Thus

$$E[\|v(\cdot,t) - v^*(\cdot,t)\|^2] = 0,$$

So that,

$$v(x,t) = v^*(x,t), almostsurely,$$

But v and v^* have continuous trajectories almost surely, so

$$P\left[\max_{0\leq t\leq T} \|v_n(\cdot,t) - v_{n-1}(\cdot,t)\|^2 > 0\right] = 0.$$

This completes the proof of the existence and uniqueness of the solution of equation (2.10), (see [6-11].

Set

$$\phi_n(x) = \int G(x-y, \frac{1}{n})\phi(y)dy.$$

It is clear that

$$\lim_{n\to\infty} E[\|\phi_n - \phi\|^2] = 0.$$

Let:

$$u_n(x,t) = \int G\left(x-y, \frac{1}{n} - \frac{t}{nT}\right) v(y, \frac{t}{nT})dy.$$

for every n, u_n solves equation (1.1) with the initial condition

$$u_n(x,0) = \phi_n(x).$$

In this case it is enough to assume that $\phi \in H$.

It is clear also that u_n satisfies the conditions I- IV.

Let us try to solve the Cauchy problem (1.3) , (1.4). For this purpose, we set:

$$f_n(x,t,Bu) = \int G(x-y, \frac{1}{n})f(y,t,Bu(y,t))dy,$$
$$g_n(x,t,Bu) = \int G(x-y, \frac{1}{n})g(y,t,Bu(y,t))dy,$$

We shall prove the existence of the sequence $\{u_n\}$ such that:

$$u_n(x,t) = \phi_n(x) + \int_0^t f_n(y,t,Bu_n(y,s))ds + \int_0^t g_n(y,t,Bu_n(y,s))dW(s). \tag{3.1}$$

Set

$$v(x,t) = \int G(x-y, c_1 t)u^*(y,t)dy,$$

where

$$u^*(x,t) = \phi(x) + \int_0^t f(x,s,BV(x,s))ds + \int_0^t g(x,s,BV(x,s))dW(s),$$
$$V(x,t) = \int G(x-y,c_2)u^*(y,t)dy, c_2 > tc_1, c_1 > 0.$$

The semi group property leads to

$$V(x,t) = \int G(x-y, c_2 - c_1 t)v(y,t)dy. \tag{3.2}$$

The properties of G and the semi group property lead formally to

$$v(x,t) = \int G(x-y, c_1 t)\phi(y)dy \tag{3.3}$$
$$+\int\int_0^t G(x-y, c_1 t)f(y,s,BV(y,s))dyds \tag{3.4}$$
$$+\int\int_0^t G(x-y, c_1 t)g(y,s,BV(y,s))dydW(s). \tag{3.5}$$

The existence and uniqueness of the solution of the Cauchy problem (1.3), (1.4) can be proved by the method of successive approximations. As usual we set

$$\begin{aligned} v_{n+1}(x,t) &= \int G(x-y, c_1 t)\phi(y)dy \\ &+\int\int_0^t G(x-y, c_1 t)f(y,s,BV_n(y,s))dyds \\ &+\int\int_0^t G(x-y, c_1 t)g(y,s,BV_n(y,s))dydW(s), \end{aligned}$$

where

$$V_n(x,t) = \int G(x-y, c_2 - c_1 t)v_n(y,t)dy.$$

The zero approximation v_0 is given by

$$v_0((x,t) = \int G(x-y, c_1 t)\phi(y)dy.$$

Using conditions (1.7) and (1.8), one gets

$$E\left[\|v_1(\cdot,t) - v_0(\cdot,t)\|^2\right] \leq Kt.$$

Using (1.5) , (1.6), (3.2) and similar methods to the linear case, we can prove the existence and uniqueness of v. Let

$$c_2 = \frac{1}{n}, c_1 = \frac{1}{Tn}, so$$

$$u_n(x,t) = \int G\left(x - y, \frac{1}{n} - \frac{t}{nT}\right) v(y,t)dy.$$

It is clear that, for every n ,$u_n(x,t)$ solves equation (3.1), (Comp [12-20]).

4 Conclusion

Using a parapolic transfom, we have got suitable results on the correct formulation of the Cauchy problem for general stochastic partial differential equations. With the help of the parabolic transform, we can solve the Cauchy problem for general stochatic parial differential equatins without any restrictions on the charectristic forms.

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Competing Interests

The authors declare that no competing interests exist.

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