Journal of Advances in Mathematics and Computer Science

Volume 38, Issue 7, Page 122-132, 2023; Article no.JAMCS.99620
ISSN: 2456-9968
(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Research on Equation $\varphi(x)+2=\varphi(x+2)$

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2023/v38i71778

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here:
https://www.sdiarticle5.com/review-history/99620

Original Research Article

Received: 01/03/2023
Accepted: 03/05/2023
Published: 15/05/2023

Abstract

In this paper, we use the properties of Euler's function, elementary methods and the idea of classification discussion to study the solvability of equation $\varphi(x)+2=\varphi(x+2)$.

Keywords: Euler function; equation; solution; Mersenne prime; Twin prime.

1 Introduction

Research on Euler's function is a very important and meaningful topic in number theory. Many scholars have studied its properties and obtained many interesting results.

Euler's function is defined as the number of positive integers that are less than or equal to *n* and relatively prime to *n*, denoted as $\varphi(n)$ [1]. From the definition, we can see that $\varphi(1)=1,\varphi(2)=1,\varphi(3)=2,\cdots$. For a prime number p, all positive integers less than p are relatively prime to p, so $\varphi(p)=p-1$. If $n>1$, let

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canonical form of n be $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$, where $p_1, p_2, ..., p_k$ are different primes, $r_i \ge 1$ $(1 \le i \le k)$, then

$$\varphi(n) = n(1-\frac{1}{p_1})(1-\frac{1}{p_2})\cdots(1-\frac{1}{p_k}) \text{ [2].}$$

Carmichael [3] proof that if $\varphi(n) = 2j$ ($j > 1$ and j is odd), then $n = p^\alpha$ or $2p^\alpha$, p is an odd prime .

In 1945，Paul Erdös [4] conjectured that the equation

$$\varphi(n) = \varphi(n+1) = \varphi(n+2) = \cdots = \varphi(n+q)$$

is solvable for arbitrary positive integer q .

For $k < 30$ and n in the range of 10^4 to 10^5, Lal and Gillard [5] provided the number of solutions for the equation $\varphi(n) = \varphi(n+k)$. Schinzel [6,7] conjecture that for every even k, the equation $\varphi(n) = \varphi(n+k)$ has infinitely many solutions, but observes that the corresponding conjecture with k odd is implausible.

Makowski [8] considered the solution of equation $\varphi(x) + \varphi(k) = \varphi(x+k)$

Patricia Jones [9] proof that if $\varphi(x) + \varphi(3) = \varphi(x+3)$, then

(i) $x = 2p^\alpha$ or $x = 2p^\alpha - 3$, and prime $p > 3$.

(ii) Either x or $x+3$ has at least 33 distinct prime factors.

(iii) $x = 2p^\alpha$, where α is odd，prime $p \equiv 2 \pmod 3$, $x > 10^{11}$, and $x+3$ has at least 9 distinct prime factors.

V. L.klee [10] listed the values of the Euler function for n < 3000, and find that the equation $\varphi(n) + 2 = \varphi(n+2)$ holds when both n and $n+2$ are prime, or n is the form of $4p$ and both p and $2p+1$ are prime.

Moser Leo [11] proof that if $\varphi(n) + 2 = \varphi(n+2)$, then at least one of n and $n+2$ is of the form p^α or $2p^\alpha$, and p is a prime number in the form of $4r+3$.

When $x > 2$, $\varphi(x)$ must be even. When $k = 1$, the equation $\varphi(x) + k = \varphi(x+k)$ has only one solution, which is $x = 2$ obviously. For odd k, it is easy to show that the equation $\varphi(x) + k = \varphi(x+k)$ has only one solution $x = 2$ when $k+2$ is prime. For even k, it is more difficult, we study the equation $\varphi(x) + k = \varphi(x+k)$ due to $k = 2$, and get the following results.

Theorem 1: If prime $p \equiv 3 \pmod 4$ and positive integer α satisfying

$$\varphi(2p^\alpha - 2) + 2 = \varphi(2p^\alpha),$$

then α = 1 and both p and $\frac{p-1}{2}$ are primes.

Theorem 2: If prime $p \equiv 3 \pmod 4$ and positive integer α satisfying $\varphi(2p^{\alpha})+2=\varphi(2p^{\alpha}+2)$, except for the cases when $\alpha=1$ and p is a Mersenne prime, or when $\alpha=2$ and $p=3$, any other solutions must satisfy the following $\alpha=2^{a}\,(a>1)$ and $\omega\left(\frac{p^{2^{a}}+1}{2}\right)>1$, where $\omega(n)$ denotes the number of distinct prime factors of n.

Theorem 3: If prime $p \equiv 3 \pmod 4$ and positive integer α satisfying $\varphi(p^{\alpha})+2=\varphi(p^{\alpha}+2)$, except for the cases when $\alpha=1$, both p and $p+2$ are prime, any other solutions must satisfy the following conditions, $\alpha>1$ is odd, $p \equiv 11 \pmod{12}$ and p^{α} has one prime factor $q \equiv 1 \pmod 3$ at least.

Theorem 4: If prime $p \equiv 3 \pmod 4$ and positive integer α satisfying $\varphi(p^{\alpha}-2)+2=\varphi(p^{\alpha})$, except for the cases when $\alpha=1$, p and $p-2$ are twin primes, any other solutions must satisfy the following conditions:

(1) $p=3$ and $3^{\alpha}-2$ has even number of prime factors $q \equiv 2 \pmod 3$ or
(2) $p \equiv 1 \pmod 3$ and $p^{\alpha}-2$ has odd number of prime factors $q \equiv 2 \pmod 3$ or
(3) $p \equiv 2 \pmod 3$, α is even and $p^{\alpha}-2$ has one prime factor $q \equiv 1 \pmod 3$ at least .

Theorem 5: Except for $x=18$ or $x=2M_p$ where M_p is a Mersenne prime, or x and $x+2$ are twin primes, or $x=2p-2$ and both p and $\frac{p-1}{2}$ are primes, other solutions of the equation $\varphi(x)+2=\varphi(x+2)$ must satisfy one of the following conditions:

(i) $x=2p^{\alpha}$,where $\alpha=2^{a}\,(a>1)$ and $\omega\left(\frac{p^{2^{a}}+1}{2}\right)>1$.

(ii) $x=p^{\alpha}$, where $\alpha>1$ is odd, $p \equiv 11 \pmod{12}$ and p^{α} has one prime factor $q \equiv 1 \pmod 3$ at least .

(iii) $x=3^{\alpha}-2$ and all the prime factors of x must be the form $3r+2$.

(iv) $x=p^{\alpha}-2$, where either $p \equiv 1 \pmod 3$ and all the prime factors of x must be the form $3r+2$ or $p \equiv 2 \pmod 3$, α is even and x has one prime factor $q \equiv 1 \pmod 3$ at least.

2 Preliminaries

Lemma 1 [12]: If n is an odd integer, then $\varphi(2n)=\varphi(n)$. If n is an even integer, then $\varphi(2n)=2\varphi(n)$.

Lemma 2 [12]: If $\varphi(n)=n-1$, then n is a prime.

Lemma 3 [11]: If $\varphi(n)+2=\varphi(n+2)$ then at least one of n and n+2 has the form p^{α} or $2p^{\alpha}$, where p is a prime of the form 4 r +3.

Lemma 4: If $\varphi(n)=\frac{n}{2}$, then $n=2^{\alpha}\,(\alpha>0)$.

Proof: Let $n=2^{\alpha}n_1\left(\alpha>0,(2,n_1)=1\right)$. Then

$$2^{\alpha-1}n_1=\frac{n}{2}=\varphi(n)=\varphi\left(2^{\alpha}\right)\varphi(n_1)=2^{\alpha-1}\varphi(n_1).$$

So $\varphi(n_1)=n_1$, we have $n_1=1$. Thus $n=2^{\alpha}\,(\alpha>0)$.

3 Proof of the Theorems

3.1 Proof of theorem 1

For the equation $\varphi\left(2p^{\alpha}-2\right)+2=\varphi\left(2p^{\alpha}\right)$, since $p\equiv 3(\mathrm{mod}\,4)$ and α is positive integer, then $p^{\alpha}-1$ is even. By Lemma 1, we have

$$p^{\alpha}-p^{\alpha-1}-2=2\varphi\left(p^{\alpha}-1\right).$$

(1) When $\alpha=1$, $p-3=2\varphi(p-1)$, it is obviously that $p\neq 3$. Since p is a prime number of the form $4r+3$, $\frac{p-1}{2}$ is an odd prime. Therefore, by Lemma 1, we have

$$\varphi\left(\frac{p-1}{2}\right)=\frac{p-3}{2}=\frac{p-1}{2}-1.$$

By Lemma 2, we have $\frac{p-1}{2}$ is a prime.

(2) When $\alpha>1$ is odd, there exists a positive integer M such that

$$p^{\alpha}-1=(p-1)M .$$

Also, since p is a prime of the form $4r+3$, it follows that $\frac{p-1}{2}$ is odd.Thus

$$2p^{\alpha}-2=2(p-1)M=4\cdot\frac{p-1}{2}\cdot M \quad \text{period.}$$

Therefore, $2p^{\alpha}-2$ must have an odd prime factor not exceeding $\frac{p-1}{2}$, so

$$\varphi\left(2p^{\alpha}-2\right)\leq\left(2p^{\alpha}-2\right)\left(1-\frac{1}{2}\right)\left(1-\frac{2}{p-1}\right).$$

Furthermore $\varphi\left(2p^{\alpha}-2\right)=\varphi\left(2p^{\alpha}\right)-2=p^{\alpha}-p^{\alpha-1}-2$, so

$$p^{\alpha}-p^{\alpha-1}-2\le\left(2p^{\alpha}-2\right)\left(1-\frac{1}{2}\right)(1-\frac{2}{p-1}).$$

Hence $\left(p^{\alpha-1}-1\right)\left(p+1\right)\le 0$, it is impossible.

(3) When α is even, if $p>3$, since $p^{\alpha}\equiv 1\left(\bmod 2\right), p^{\alpha}\equiv 1\left(\bmod 3\right)$,

$$p^{\alpha}-p^{\alpha-1}-2=2\varphi\left(p^{\alpha}-1\right)\le 2\left(p^{\alpha}-1\right)\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right).$$

Hence $p^{\alpha}-3p^{\alpha-1}-4<0$, i.e., $p<4$ is contradictory to condition $p>3$.

If $p=3$, then $\varphi\left(2\cdot 3^{\alpha}\right)=\varphi\left(2\cdot 3^{\alpha}-2\right)+2$, by Lemma 1, we have

$$\varphi\left(3^{\alpha}\right)=2\varphi\left(3^{\alpha}-1\right)+2, \text{ i.e., } 3^{\alpha-1}-1=\varphi\left(3^{\alpha}-1\right).$$

As $3^{\alpha}-1\equiv 0\left(\bmod 8\right)$, according to the computation and properties of Euler's totient function, we can obtain $\varphi\left(3^{\alpha}-1\right)\equiv 0\left(\bmod 4\right)$, but $3^{\alpha-1}-1\equiv 2\left(\bmod 4\right)$, it is contradictory.

Combining with (1)、(2) and (3), we obtain the conclusion of Theorem 1.

3.2 Proof of theorem 2

For the equation $\varphi\left(2p^{\alpha}\right)+2=\varphi\left(2p^{\alpha}+2\right)$, since $\varphi\left(2p^{\alpha}\right)=p^{\alpha}-p^{\alpha-1}$, we have

$$\varphi\left(2p^{\alpha}+2\right)=p^{\alpha}-p^{\alpha-1}+2. \tag{3.1}$$

(1) When $\alpha=1$, by Lemma 1, we obtain

$$\varphi\left(p+1\right)=\frac{p+1}{2}.$$

By Lemma 4, we have $p+1=2^{\beta}$,that is $p=2^{\beta}-1$ is a Mersenne prime.

(2) When $\alpha>1$ is odd, there exists a positive integer M such that

$$2p^{\alpha}+2=4\cdot\frac{p+1}{2}\cdot M.$$

If $p+1=2^{k}$, then $2p^{\alpha}+2=2^{k+1}\cdot\frac{p^{\alpha}+1}{p+1}$. And

$$\frac{p^{\alpha}+1}{p+1}=p^{\alpha-1}-p^{\alpha-2}+p^{\alpha-3}-p^{\alpha-4}+\cdots+1>1$$

is odd. Therefore, the left side of (3-1) is

$$\varphi\left(2p^{\alpha}+2\right)=\varphi(2^{k+1})\varphi\left(\frac{p^{\alpha}+1}{p+1}\right)=2^{k}\varphi\left(\frac{p^{\alpha}+1}{p+1}\right)\equiv 0\left(\mathrm{mod}\,2^{k+1}\right).$$

But, the right side of (3-1) is

$$\begin{aligned}
&p^{\alpha}-p^{\alpha-1}+2=p^{\alpha-1}\left(p-1\right)+2=\left(2^{k}-2\right)\left(2^{k}-1\right)^{\alpha-1}+2\\
&=\left(2^{k}-2\right)\left(C_{\alpha-1}^{0}2^{k(\alpha-1)}+C_{\alpha-1}^{1}2^{k(\alpha-2)}\left(-1\right)+\cdots+C_{\alpha-1}^{\alpha-2}2^{k}\left(-1\right)^{\alpha-2}+1\right)+2\\
&=2^{k}\left(C_{\alpha-1}^{0}2^{k(\alpha-1)}+C_{\alpha-1}^{1}2^{k(\alpha-2)}\left(-1\right)+\cdots+C_{\alpha-1}^{\alpha-2}2^{k}\left(-1\right)^{\alpha-2}\right)\\
&-2\left(C_{\alpha-1}^{0}2^{k(\alpha-1)}+C_{\alpha-1}^{1}2^{k(\alpha-2)}\left(-1\right)+\cdots+C_{\alpha-1}^{\alpha-2}2^{k}\left(-1\right)^{\alpha-2}\right)+2^{k}\\
&\equiv 2^{k}\left(\mathrm{mod}\,2^{k+1}\right).
\end{aligned}$$

Contradictory, thus $p+1\neq 2^{k}$. So $p^{\alpha}+1$ must have an odd prime factor not exceeding $\frac{p+1}{4}$. Thus

$$p^{\alpha}-p^{\alpha-1}+2=\varphi\left(2p^{\alpha}+2\right)\leq\left(2p^{\alpha}+2\right)\left(1-\frac{1}{2}\right)\left(1-\frac{p+1}{4}\right),$$

hence $3p^{\alpha}-p^{\alpha-1}+p+5\leq 0$, it is impossible.

(3) When $\alpha=2^{a}b$, $(2,b)=1$, if $b>1$, then

$$p^{2^{a}b}-p^{2^{a}b-1}+2=\varphi\left(2p^{2^{a}b}+2\right),$$

because $2p^{2^{a}b}+2$ must have factors 2 and $p^{2^{a}}+1$,

$$\varphi\left(2p^{2^{a}b}+2\right)\leq\left(2p^{2^{a}b}+2\right)(1-\frac{1}{2})\left(1-\frac{2}{p^{2^{a}}+1}\right)=\left(p^{2^{a}b}+1\right)\frac{p^{2^{a}}-1}{p^{2^{a}}+1}.$$

Hence $p^{2^{a}}+p^{2^{a}b}-p^{2^{a}b-1}+3<0$, it is impossible. Therefore, if such an even number α exists, then it must be $b=1$ and $\alpha=2^{a}(a\geq 1)$. In this case (3-1) is

$$\varphi\left(2p^{2^{a}}+2\right)=p^{2^{a}}-p^{2^{a}-1}+2.$$

Since p is an odd prime, $p^{2^{a}}+1\equiv 2\ (\mathrm{mod}\ 8)$, thus $\frac{p^{2^{a}}+1}{2}$ is odd, by Lemma 1, we have

$$\varphi\left(\frac{p^{2^a}+1}{2}\right)=\frac{p^{2^a}-p^{2^a-1}+2}{2}. \tag{3.2}$$

Since

$$\left(\frac{p^{2^a}+1}{2},\frac{p^{2^a}-p^{2^a-1}+2}{2}\right)=\left(\frac{p^{2^a}+1}{2},\frac{p^{2^a-1}-1}{2}\right)=\left(\frac{p+1}{2},\frac{p^{2^a-1}+1}{2}-1\right)=\left(\frac{p+1}{2},1\right)=1,$$

$\frac{p^{2^a}+1}{2}$ is square-free.

(i) When $\omega\left(\frac{p^{2^a}+1}{2}\right)=1$, (3-2) is

$$\frac{p^{2^a}+1}{2}-1=\frac{p^{2^a}-p^{2^a-1}+2}{2},$$

it leads to $p^{2^a-1}=3$, so $p=3, a=1$, in this case $x=2p^{2^a}=2\times 3^2=18.$

(ii) When $\omega\left(\frac{p^{2^a}+1}{2}\right)>1$, then $\frac{p^{2^a}+1}{2}$ must have an odd prime factor that is less than $\sqrt{\frac{p^{2^a}+1}{2}}$, so

$$\varphi\left(\frac{p^{2^a}+1}{2}\right)<\left(\frac{p^{2^a}+1}{2}\right)\left(1-\frac{1}{\sqrt{\frac{p^{2^a}+1}{2}}}\right)=\frac{p^{2^a}+1}{2}-\sqrt{\frac{p^{2^a}+1}{2}}.$$

By (3-2)

$$\frac{p^{2^a}-p^{2^a-1}+2}{2}<\frac{p^{2^a}+1}{2}-\sqrt{\frac{p^{2^a}+1}{2}},$$

it leads $2p^{2^a}-p^{2^{a+1}-2}+2p^{2^a-1}+1<0$. When $a=1$, $p^2+2p+1<0$, it is impossible.

Therefore, if such an α exists, then it must be $\alpha=2^a\left(a>1\right)$ and $\omega\left(\frac{p^{2^a}+1}{2}\right)>1.$

Combining with (1)、(2) and (3), we obtain the conclusion of Theorem 2.

3.3 Proof of theorem 3

For $\varphi\left(p^{\alpha}+2\right)=\varphi\left(p^{\alpha}\right)+2=p^{\alpha}-p^{\alpha-1}+2$, since

$$\left(p^{\alpha}+2,\ p^{\alpha}-p^{\alpha-1}+2\right)=\left(p^{\alpha}+2,-p^{\alpha-1}\right)=1,$$

$p^{\alpha}+2$ is square-free.

(1) When $\alpha=1$, $\varphi(p+2)=p+1$, by Lemma 2, we have $p+2$ is prime, that is when p and $p+2$ are twin primes, the equation $\varphi(p^{\alpha}+2)=\varphi(p^{\alpha})+2$ holds.

(2) When $\alpha>1$ and $p=3$, $\varphi(3^{\alpha}+2)=3^{\alpha}-3^{\alpha-1}+2$, let $3^{\alpha}+2=q_1q_2\cdots q_i$. Since $3^{\alpha}+2\equiv 2(\bmod 3)$, $3^{\alpha}+2$ must have an odd number of prime factors $q_j\equiv 2(\bmod 3)\,(1\le j\le i)$.

(i) If all $q_j\equiv 2(\bmod 3)\,(1\le j\le i)$, then

$$\varphi(3^{\alpha}+2)=(q_1-1)(q_2-1)\cdots(q_i-1)\equiv 1(\bmod 3),$$

(ii) If there exists $q_j\equiv 1(\bmod 3)\,(1\le j\le i)$, then

$$\varphi(3^{\alpha}+2)=(q_1-1)(q_2-1)\cdots(q_i-1)\equiv 0(\bmod 3),$$

but $3^{\alpha}-3^{\alpha-1}+2\equiv 2(\bmod 3)$, Contradiction.

(3) When $p>3$ and α is even, since $3\mid(p^{\alpha}+2)$, we have

$$p^{\alpha}-p^{\alpha-1}+2=\varphi(p^{\alpha}+2)<(p^{\alpha}+2)\left(1-\frac{1}{3}\right).$$

That is $p^{\alpha-1}(p-3)+2<0$, contradiction.

(4) When $p>3$ and $\alpha>1$ is odd

(i) If $p\equiv 1(\bmod 3)$, then $3\mid p^{\alpha}+2$, so $p^{\alpha}+2$ must have a factor 3, thus

$$p^{\alpha}-p^{\alpha-1}+2=\varphi(p^{\alpha}+2)<(p^{\alpha}+2)\left(1-\frac{1}{3}\right),$$

Simplifying gives $p^{\alpha-1}(p-3)+2<0$, contradiction.

(ii) If $p\equiv 2(\bmod 3)$, then $p^{\alpha}+2\equiv 1(\bmod 3)$. Let $p^{\alpha}+2=q_1q_2\cdots q_i$.

If all $q_j\equiv 2(\bmod 3)\,(1\le j\le i)$, then

$$\varphi(p^{\alpha}+2)=(q_1-1)(q_2-1)\cdots(q_i-1)\equiv 1(\bmod 3).$$

But $p^{\alpha}-p^{\alpha-1}+2\equiv 0(\bmod 3)$, contradiction. Thus there must at least exist one prime $q_j\equiv 1(\bmod 3)\,(1\le j\le i)$. Furthermore $p\equiv 3(\bmod 4)$, by Chinese Remainder Theorem, we have $p\equiv 11(\bmod 12)$ and $p^{\alpha}+2$ has at least one prime factor $q\equiv 1(\bmod 3)$.

Combining with (1)、(2)、(3) and (4), we obtain the conclusion of Theorem 3.

3.4 Proof of theorem 4

For the equation $\varphi\left(p^{\alpha}-2\right)+2=\varphi\left(p^{\alpha}\right)$ we have

$$\varphi\left(p^{\alpha}-2\right)=p^{\alpha}-p^{\alpha-1}-2. \tag{3.3}$$

(1) When $\alpha=1$, (3-3) is $\varphi(p-2)=p-3$. By Lemma 2, we have $p-2$ is prime, so when p and $p-2$ are twin primes, (3-3) holds.

(2) When $\alpha>1$ and $p=3$, (3-3) is $\varphi\left(3^{\alpha}-2\right)=3^{\alpha}-3^{\alpha-1}-2$.

Let $3^{\alpha}-2=q_1q_2\cdots q_i$. If there exists a prime factor $q_j\equiv1(\operatorname{mod}3)$ $(1\le j\le i)$, then $\varphi\left(3^{\alpha}-2\right)\equiv0(\operatorname{mod}3)$, but $3^{\alpha}-3^{\alpha-1}-2\equiv1(\operatorname{mod}3)$, contradiction.

Since $3^{\alpha}-2\equiv1(\operatorname{mod}3)$, then we have all prime factors $q_j\equiv2(\operatorname{mod}3)$ $(1\le j\le i)$ and i is even.

(3) When $\alpha>1$ and $p\equiv1(\operatorname{mod}3)$, we have $p^{\alpha}-2\equiv2(\operatorname{mod}3)$, let $p^{\alpha}-2=\prod_{j=1}^{i}q_j$.

If there exists a prime factor $q_j\equiv1(\operatorname{mod}3)$ $(1\le j\le i)$, then $\varphi\left(p^{\alpha}-2\right)\equiv0(\operatorname{mod}3)$, but $p^{\alpha}-p^{\alpha-1}-2\equiv1(\operatorname{mod}3)$, contradiction.

Thus all prime factors of $p^{\alpha}-2$ satisfying $q_j\equiv2(\operatorname{mod}3)$ $(1\le j\le i)$ and i is odd.

(4) When $\alpha>1$ and $p\equiv2(\operatorname{mod}3)$.

(i) If α is odd, then $3\mid(p^{\alpha}-2)$, therefore

$$p^{\alpha}-p^{\alpha-1}-2=\varphi\left(p^{\alpha}-2\right)<\left(p^{\alpha}-2\right)\left(1-\frac{1}{3}\right),$$

it gives $p^{\alpha-1}(p-3)<2$, contradiction.

(ii) If α is even, then $p^{\alpha}-2\equiv2(\operatorname{mod}3)$, let $p^{\alpha}-2=q_1q_2\cdots q_i$.

If all $q_j\equiv2(\operatorname{mod}3)$ $(1\le j\le i)$, then $\varphi\left(p^{\alpha}-2\right)\equiv1(\operatorname{mod}3)$, but
$p^{\alpha}-p^{\alpha-1}-2\equiv0(\operatorname{mod}3)$,

contradiction. Thus there exists one prime factor of $p^{\alpha}-2$ satisfying $q_j\equiv1(\operatorname{mod}3)$.

Combining with (1)、(2)、(3) and (4), we obtain the conclusion of Theorem 4.

3.5 Proof of theorem 5

By Lemma 3, we know that the solution of equation $\varphi(x)+2=\varphi(x+2)$ satisfies $x=2p^{\alpha}-2,\ p^{\alpha}-2,\ p^{\alpha}$ or $2p^{\alpha}$ and $p\equiv 3(\bmod 4)$, α is a positive integer . Based on Theorem 1-4, we can conclude that the solutions to the equation

$$\varphi(x)+2=\varphi(x+2)$$

satisfying one of the following:

(1) $x=2p-2$, both p and $\frac{p-1}{2}$ are primes;

(2) x and $x+2$ are twin primes;

(3) $x=18$ or $x=2M_p$, where M_p is a Mersenne prime;

(4) The other solutions x must satisfy one of the following

(i) $x=2p^{\alpha}$, where $\alpha=2^{a}\ (a>1)$ and $\omega\left(\frac{p^{2^{a}}+1}{2}\right)>1$.

(ii) $x=p^{\alpha}$, where $\alpha>1$ and α is odd, $p\equiv 11(\bmod 12)$ and p^{α} has at least one prime factor $q\equiv 1(\bmod 3)$.

(iii) $x=3^{\alpha}-2$, and all the prime factors of x must be the form $3r+2$.

(iv) $x=p^{\alpha}-2$, where either $p\equiv 1(\bmod 3)$ and all the prime factors of x must be the form $3r+2$ or $p\equiv 2(\bmod 3)$, α is even and $p^{\alpha}-2$ has at least one prime factor $q\equiv 1(\bmod 3)$.

4 Conclusion

From the proof of Theorem 5, we conjecture that the solution to the equation $\varphi(x)+2=\varphi(x+2)$ only exists in the first three cases. That is, except for $x=18$ or $x=2M_p$ where M_p is a Mersenne prime, or x and $x+2$ are twin primes, or $x=2p-2$ and both p and $\frac{p-1}{2}$ are primes, there are no other solutions of the equation $\varphi(x)+2=\varphi(x+2)$.We hope that readers can find a solution of $\varphi(x)+2=\varphi(x+2)$ that belongs to Case 4, or prove that there is no solution that satisfies Case 4.

Acknowledgements

This work is supported by the National Natural Science Foundation of China, Project (No. 12071421).

Competing Interests

Authors have declared that no competing interests exist.

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