Advances in Research

Volume 25, Issue 5, Page 1-13, 2024; Article no.AIR.121767
ISSN: 2348-0394, NLM ID: 101666096

Smarandache Curves of Natural Curves Pair According to Frenet Frame

Thiago Mendonça [a*], José Alan [a] and Renato Teixeira [a]

[a] *Departamento de Matemática, UFRPE, Rua Manoel de Medeiros, s/n - Dois Irmãos, Recife - PE, 52171-900, Brazil.*

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information

DOI: https://doi.org/10.9734/air/2024/v25i51131

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/121767

Received: 14/06/2024
Accepted: 17/08/2024
Published: 23/08/2024

Original Research Article

ABSTRACT

Smarandache curves play a significant role in differential geometry. They extend conventional differential geometry concepts and help uncover novel geometric properties. In this work, we define the Smarandache curves of the Natural mate of any given curve α and calculate their Frenet apparatus. As a particular case, we present the Frenet apparatus when α is a helix. Additionally, we illustrate an example for the slant helix.

Keywords: Smarandache curves; natural mate curves; Frenet apparatus.

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16.

**Corresponding author: E-mail: thiago.mendonca@ufrpe.br;*

Cite as: *Mendonca, Thiago, Jose Alan, and Renato Teixeira. 2024. "Smarandache Curves of Natural Curves Pair According to Frenet Frame". Advances in Research 25 (5):1-13. https://doi.org/10.9734/air/2024/v25i51131.*

1 INTRODUCTION

A starting point in the study of differential geometry is the analysis of curves in space. By introducing special geometric properties, it is possible to produce new curves from a given curve. In this context, there are some special curves such as Bertrand, Mannheim, Natural mate, Smarandache, involute, evolute, and pedal curves which have important applications ranging from physics to surface modeling in engineering and computer graphics [1, 2, 3, 4, 5, 6, 7, 8]. For any given unit speed curve α in $\mathbb{E}^3$ there is a unique unit speed curve $\bar{\alpha}$ whose tangent vector coincides with the principal normal vector of α, called *principal-direction curve* or *Natural mate curve* [9]. Some authors have studied this class of curves: Abdel-Baky and Naghi [10] study sweeping surfaces with Natural mate curves, Deshmukh et al [11] prove some relationships between a Frenet curve and its natural mate, Camci et al [12] extend the natural mate $\bar{\alpha}$ to sequential natural mates $\{\alpha_1, \alpha_2, \cdots, \alpha_{n_\alpha}\}$ with $\alpha_1 = \bar{\alpha}$, Mak [13] introduce the natural mate curves in a three dimensional Lie group with bi-invariant metric and give some relationships between a Frenet curve and its natural mate in this group.

In differential geometry, the Smarandache geometry has a significant role in the theory of relativity and parallel universes [14]. In Smarandache geometry, regular curves that are defined as having the position vector being a combination of the tangent, normal and binormal vectors of another regular curve are called *Smarandache curves*.

Smarandache curves are used to expand traditional concepts of differential geometry and discover new geometric properties [15]. These curves allow for detailed analyses in areas such as curve theory and surface theory. Their applications span various fields, including physics, engineering, robotics, computer graphics, and biomechanics. In physics, they are especially used in space-time geometry and general relativity theory [16]. In engineering, they find application in materials science and structural analysis. In computer graphics and robotics, they play a crucial role in motion planning and object recognition.

Special Smarandache curves have been studied by some authors. Savas et al [17] studied special Smarandache curves according to the Sabban frame in hyperbolic space and new Smarandache partners in de Sitter space, Ali [18] studied some special Smarandache curves in the Euclidean space, Şenyurt et al [19] studied the Smarandache curves of Bertrand curves pair according to Frenet frame and expresses its curvature and torsion in terms of the curvature and torsion of the Bertrand curve and Şenyurt and Kaya studied the NC- Samarandache curve and NW-Smarandache curve according to alternative frame [20]. These curves have been also studied widely [21, 22, 23, 24, 25].

In this paper, we start by mentioning the Frenet apparatus of any regular curve α parametrized by arc-length in Euclidean space $\mathbb{E}^3$. Then, we relate the Frenet frame of the Natural pair and we give the definitions of the $tn, tb, nb, tnb-$Smarandache curves in $\mathbb{E}^3$ for Natural mate curves and calculate the Frenet apparatus of these curves using the curvature, torsion, the unit Darboux vector and the function angle between the Darboux vector and the binormal vector of α. We calculate the Frenet apparatus for the case where α is a helix and close with an example where α is a slant helix.

2 PRELIMINARIES

Consider the Euclidean space $\mathbb{E}^3$ with inner product

$$\langle \cdot, \cdot \rangle = dx^2 + dy^2 + dz^2,$$

where $(x, y, z) \in \mathbb{E}^3$ is a rectangular coordinate system. Let $\alpha : I \to \mathbb{E}^3$ be a differentiable curve in the Euclidean space defined on an open interval I, parametrized by arc-length and let $\{t = \alpha', n, b\}$ be Frenet frame satisfying [15]

$$\begin{cases} t' = \kappa n, \\ n' = -\kappa t + \tau b, \\ b' = -\tau n, \end{cases} \tag{2.1}$$

where κ and τ are differentiable functions on I called the *curvature* and the *torsion* of α, respectively, t is the tangent vector, n is the principal normal vector and b is the binormal vector of α. The 5-uple (t, n, b, κ, τ) is called a *Frenet apparatus*.

Definition 2.1. *A curve $\bar{\alpha} : J \to \mathbb{E}^3$ is called Natural mate curve of $\alpha : I \to \mathbb{E}^3$ if $\bar{\alpha}(\bar{s})$ is the integral curve of the principal normal vector of $\alpha(s^*)$, and the pair $(\alpha(s^*), \bar{\alpha}(\bar{s}))$ is called the Natural pair [11].*

The existence of Natural mate curves is guaranteed by existence theorem for differential equation and $\bar{\alpha}$ is given by $\bar{\alpha} = \int n ds^*$. It can be also shown that the arc-lenght parameters of the curve $\bar{\alpha}$ can be the same of α, that is, $s^* = \bar{s}$ [26].

The relations between the Frenet frames $\{t, n, b\}$ and $\{\bar{t}, \bar{n}, \bar{b}\}$ are given by [10]

$$\begin{cases} \bar{t} &= n, \\ \bar{n} &= -\cos\psi t + \sin\psi b, \\ \bar{b} &= \sin\psi t + \cos\psi b, \end{cases} \tag{2.2}$$

where $\psi = \angle(\bar{b}, b)$,

$$\cos\psi = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}} \quad \text{and} \quad \sin\psi = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}. \tag{2.3}$$

Let the Darboux vector defined by

$$W = \tau t + \kappa b.$$

The Darboux vector represents the Frenet frame's angular momentum. Its direction determines the frame's momentary axis of motion (its centroid) and its length the angular speed, $\| W \| = \sqrt{\kappa^2+\tau^2}$.
If we consider the normalization of the Darboux $C = \frac{1}{\| W \|} W$ we have,

$$\sin\varphi = \frac{\tau}{\| W \|} = \frac{\tau}{\sqrt{\kappa^2+\tau^2}} \quad \text{and} \quad \cos\varphi = \frac{\kappa}{\| W \|} = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}, \tag{2.4}$$

where $\varphi = \angle(W, b)$. Therefore, by the equations (2.3) and (2.4), we have $\psi = \varphi$, and consequently, the binormal vector of the Natural mate curve $\bar{\alpha}$ is equal to the Darboux vector of the curve α.

From equation (2.4), we obtain

$$\varphi' = \frac{\kappa\tau' - \kappa'\tau}{\| W \|^2} = \left(\frac{\tau}{\kappa}\right)' \frac{\kappa^2}{\| W \|^2}. \tag{2.5}$$

Remark 2.1. *A regular curve α is called a helix if the tangent lines of the curve make a constant angle with a fixed direction and a helix is characterized by the property that $\frac{\tau}{\kappa}$ is constant. From equation (2.5), we have in this case $\varphi =$ constant.*

3 SMARANDACHE CURVES OF NATURAL CURVES PAIR ACCORDING TO FRENET FRAME

In this section, we investigate the Smarandache curves of Natural pair according to Frenet frame in Euclidean 3-space and we give the Frenet apparatus for these curves.

Definition 3.1. *A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve [27].*

In the light of the above definition, Ali adapt it to regular curves in the Euclidean space the definition of Smarandache curves [18]. For Natural pair we have

Definition 3.2. *Let $(\alpha, \bar{\alpha})$ be a Natural pair in $\mathbb{E}^3$ and $\{\bar{t}, \bar{n}, \bar{b}\}$ be the Frenet frame of the Natural mate curve $\bar{\alpha}$. The $\bar{t}\bar{n}-$ Smarandache curve is defined by*

$$\beta_1(s) = \frac{1}{\sqrt{2}}(\bar{t}(\bar{s}) + \bar{n}(\bar{s})). \tag{3.6}$$

Theorem 3.1. *The Frenet apparatus of the $\bar{t}\bar{n}-$Smarandache curve is given by*

$$\begin{aligned}
T_{\beta_1} &= \frac{(\varphi' \sin\varphi - \kappa)t - \| W \| n + (\varphi' \cos\varphi + \tau)b}{\sqrt{(\varphi')^2 + 2 \| W \|^2}}, &(3.7)\\
N_{\beta_1} &= \frac{\lambda_1 t + \lambda_2 n + \lambda_3 b}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}, &(3.8)\\
B_{\beta_1} &= \frac{\sigma_1 t + \sigma_2 n + \sigma_3 b}{\sqrt{(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)((\varphi')^2 + 2 \| W \|^2)}}, &(3.9)\\
\kappa_{\beta_1} &= \frac{\sqrt{2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}}{[(\varphi')^2 + 2 \| W \|^2]^2}, &(3.10)\\
\tau_{\beta_1} &= \frac{\sqrt{2}[(\varphi')^2 + 2 \| W \|^2](\sigma_1\eta_1 + \sigma_2\eta_2 + \sigma_3\eta_3)}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}, &(3.11)
\end{aligned}$$

where $W = \tau t + \kappa b$ and the coefficients are given by

$$\begin{aligned}
\lambda_1 &= \left[(\varphi')^2 \cos\varphi - \kappa' + \kappa \| W \|\right] \cdot \left[(\varphi')^2 + 2 \| W \|^2\right] + 2\varphi'' \| W \| \tau \\
&+ \kappa(\varphi'\varphi'' + 2 \| W \| \| W \|'), \\
\lambda_2 &= - \| W \|^2 \cdot [(\varphi')^2 + 2 \| W \|^2] + \varphi'(\| W \| \varphi'' - \| W \|' \varphi'), \\
\lambda_3 &= [-(\varphi')^2 \sin\varphi + \tau' - \tau \| W \|] \cdot [(\varphi')^2 + 2 \| W \|^2] + 2\varphi'' \| W \| \kappa \\
&- \tau(\varphi'\varphi'' + 2 \| W \| \| W \|'),
\end{aligned}$$

$$\begin{aligned}
\sigma_1 &= -\lambda_2(\varphi' \cos\varphi + \tau) - \lambda_3 \| W \|, \\
\sigma_2 &= \lambda_1(\varphi' \cos\varphi + \tau) - \lambda_3(\varphi' \sin\varphi - \kappa), \\
\sigma_3 &= \lambda_1 \| W \| + \lambda_2(\varphi' \sin\varphi - \kappa),
\end{aligned}$$

$$\begin{aligned}
\eta_1 &= \left(\frac{\lambda_1}{[(\varphi')^2 + 2 \| W \|^2]^2}\right)' - \frac{\kappa\lambda_2}{[(\varphi')^2 + 2 \| W \|^2]^2}, \\
\eta_2 &= \left(\frac{\lambda_2}{[(\varphi')^2 + 2 \| W \|^2]^2}\right)' + \frac{\kappa\lambda_1 - \tau\lambda_3}{[(\varphi')^2 + 2 \| W \|^2]^2}, \\
\eta_3 &= \left(\frac{\lambda_3}{[(\varphi')^2 + 2 \| W \|^2]^2}\right)' + \frac{\tau\lambda_2}{[(\varphi')^2 + 2 \| W \|^2]^2}.
\end{aligned}$$

Proof. Substituting the equation (2.2) into equation (3.6), we obtain

$$\beta_1 = \frac{-\cos\varphi t + n + \sin\varphi b}{\sqrt{2}}. \tag{3.12}$$

Taking the derivative of the equation (3.12) with respect to $\bar{s}$, we get

$$T_{\beta_1}\frac{ds}{d\bar{s}} = \frac{(\varphi' \sin\varphi - \kappa)t - \| W \| n + (\varphi' \cos\varphi + \tau)b}{\sqrt{2}}.$$

Thus

$$T_{\beta_1} = \frac{(\varphi' \sin\varphi - \kappa)t- \parallel W \parallel n + (\varphi' \cos\varphi + \tau)b}{\sqrt{(\varphi')^2 + 2 \parallel W \parallel^2}}, \tag{3.13}$$

where

$$\frac{ds}{d\bar{s}} = \frac{\sqrt{(\varphi')^2 + 2 \parallel W \parallel^2}}{\sqrt{2}}. \tag{3.14}$$

Taking the derivative of the equation (3.13) with respect to $\bar{s}$ and use (3.14), we obtain

$$T'_{\beta_1} = \frac{\sqrt{2}(\lambda_1 t + \lambda_2 n + \lambda_3 b)}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2},$$

where

$$\begin{aligned}
\lambda_1 &= ((\varphi')^2 \cos\varphi - \kappa' + \kappa \parallel W \parallel)[(\varphi')^2 + 2 \parallel W \parallel^2] + 2\varphi'' \parallel W \parallel \tau \\
&+ \kappa(\varphi'\varphi'' + 2 \parallel W \parallel \parallel W \parallel'), \\
\lambda_2 &= - \parallel W \parallel^2 [(\varphi')^2 + 2 \parallel W \parallel^2] + \varphi'(\parallel W \parallel \varphi'' - \parallel W \parallel' \varphi'), \\
\lambda_3 &= (-(\varphi')^2 \sin\varphi + \tau' - \tau \parallel W \parallel)[(\varphi')^2 + 2 \parallel W \parallel^2] + 2\varphi'' \parallel W \parallel \kappa \\
&- \tau(\varphi'\varphi'' + 2 \parallel W \parallel \parallel W \parallel'),
\end{aligned}$$

Therefore, the curvature, the principal normal vector and the binormal vector of the curve β_1 are given by

$$\kappa_{\beta_1} = \frac{\sqrt{2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2}, \qquad N_{\beta_1} = \frac{\lambda_1 t + \lambda_2 n + \lambda_3 b}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}},$$

$$\begin{aligned}
B_{\beta_1} &= T_{\beta_1} \times N_{\beta_1} \\
&= \frac{\sigma_1 t + \sigma_2 n + \sigma_3 b}{\sqrt{(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)((\varphi')^2 + 2 \parallel W \parallel^2)}},
\end{aligned}$$

where

$$\begin{aligned}
\sigma_1 &= -\lambda_2(\varphi' \cos\varphi + \tau) - \lambda_3 \parallel W \parallel, \\
\sigma_2 &= \lambda_1(\varphi' \cos\varphi + \tau) - \lambda_3(\varphi' \sin\varphi - \kappa), \\
\sigma_3 &= \lambda_1 \parallel W \parallel + \lambda_2(\varphi' \sin\varphi - \kappa).
\end{aligned}$$

The torsion of curve β_1 is given by

$$\tau_{\beta_1} = \frac{\det(\beta_1', \beta_1'', \beta_1''')}{\parallel \beta_1' \times \beta_1'' \parallel^2} = \frac{\sqrt{2}[(\varphi')^2 + 2 \parallel W \parallel^2](\sigma_1\eta_1 + \sigma_2\eta_2 + \sigma_3\eta_3)}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2},$$

where

$$\begin{aligned}
\eta_1 &= \left(\frac{\lambda_1}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2}\right)' - \frac{\kappa\lambda_2}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2}, \\
\eta_2 &= \left(\frac{\lambda_2}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2}\right)' + \frac{\kappa\lambda_1 - \tau\lambda_3}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2}, \\
\eta_3 &= \left(\frac{\lambda_3}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2}\right)' + \frac{\tau\lambda_2}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2}.
\end{aligned}$$

$\square$

From Remark (2.1) and Theorem (3.1), we get

Corollary 3.1. *If α is a helix, then the Frenet apparatus of the $\bar{t}\bar{n}-$Smarandache curve of its Natural mate curve is given by*

$$T_{\beta_1}=-\frac{1}{\sqrt{2}C_1}t-\frac{1}{\sqrt{2}}n+\frac{C}{\sqrt{2}C_1}b,\quad N_{\beta_1}=\frac{1}{\sqrt{2}C_1}t-\frac{1}{\sqrt{2}}n-\frac{C}{\sqrt{2}C_1}b,$$
$$B_{\beta_1}=\frac{C}{C_1}t+\frac{1}{C_1}b,\quad \kappa_{\beta_1}=1,\quad \tau_{\beta_1}=0,$$

where $\frac{\tau}{\kappa}=C=$ constant and $C_1=\sqrt{1+C^2}$.

Definition 3.3. *Let $(\alpha,\bar{\alpha})$ be a Natural pair in $\mathbb{E}^3$ and $\{\bar{t},\bar{n},\bar{b}\}$ be the Frenet frame of the Natural mate curve $\bar{\alpha}$. The $\bar{n}\bar{b}-$Smarandache curve is defined by*

$$\beta_2(s)=\frac{1}{\sqrt{2}}(\bar{n}(\bar{s})+\bar{b}(\bar{s})). \tag{3.15}$$

Theorem 3.2. *The Frenet apparatus of the $\bar{n}\bar{b}-$Smarandache curve is given by*

$$T_{\beta_2} = \frac{\varphi'(\sin\varphi+\cos\varphi)t-\parallel W\parallel n+\varphi'(\cos\varphi-\sin\varphi)b}{\sqrt{2(\varphi')^2+\parallel W\parallel^2}}, \tag{3.16}$$
$$N_{\beta_2} = \frac{\lambda_1 t+\lambda_2 n+\lambda_3 b}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}, \tag{3.17}$$
$$B_{\beta_2} = \frac{\sigma_1 t+\sigma_2 n+\sigma_3 b}{\sqrt{(\lambda_1^2+\lambda_2^2+\lambda_3^2)(2(\varphi')^2+\parallel W\parallel^2)}}, \tag{3.18}$$
$$\kappa_{\beta_2} = \frac{\sqrt{2(\lambda_1^2+\lambda_2^2+\lambda_3^2)}}{[2(\varphi')^2+\parallel W\parallel^2]^2}, \tag{3.19}$$
$$\tau_{\beta_2} = \frac{\sqrt{2}[2(\varphi')^2+\parallel W\parallel^2](\sigma_1\eta_1+\sigma_2\eta_2+\sigma_3\eta_3)}{\lambda_1^2+\lambda_2^2+\lambda_3^2}, \tag{3.20}$$

where $W=\tau t+\kappa b$ and the coefficients are given by

$$\begin{aligned}
\lambda_1 &= ((\varphi')^2(\cos\varphi-\sin\varphi)+\kappa\parallel W\parallel)[2(\varphi')^2+\parallel W\parallel^2]\\
&+\parallel W\parallel(\sin\varphi+\cos\varphi)(\varphi''\parallel W\parallel-\varphi'\parallel W\parallel'),\\
\lambda_2 &= \varphi'(\parallel W\parallel[2(\varphi')^2+\parallel W\parallel^2]+2(\varphi''\parallel W\parallel-\varphi'\parallel W\parallel')),\\
\lambda_3 &= -((\varphi')^2(\sin\varphi+\cos\varphi)+\tau\parallel W\parallel)[2(\varphi')^2+\parallel W\parallel^2]\\
&+\parallel W\parallel(\cos\varphi-\sin\varphi)(\varphi''\parallel W\parallel+\varphi'\parallel W\parallel'),
\end{aligned}$$

$$\begin{aligned}
\sigma_1 &= -\lambda_2\varphi'(\cos\varphi-\sin\varphi)-\lambda_3\parallel W\parallel,\\
\sigma_2 &= \lambda_1\varphi'(\cos\varphi-\sin\varphi)-\lambda_3\varphi'(\sin\varphi+\cos\varphi),\\
\sigma_3 &= \lambda_1\parallel W\parallel+\lambda_2\varphi'(\sin\varphi+\cos\varphi),
\end{aligned}$$

$$
\begin{aligned}
\eta_1 &= \left(\frac{\lambda_1}{[2(\varphi')^2+\parallel W\parallel^2]^2}\right)' - \frac{\kappa\lambda_2}{[2(\varphi')^2+\parallel W\parallel^2]^2},\\
\eta_2 &= \left(\frac{\lambda_2}{[2(\varphi')^2+\parallel W\parallel^2]^2}\right)' + \frac{\kappa\lambda_1-\tau\lambda_3}{[2(\varphi')^2+\parallel W\parallel^2]^2},\\
\eta_3 &= \left(\frac{\lambda_3}{[2(\varphi')^2+\parallel W\parallel^2]^2}\right)' + \frac{\tau\lambda_2}{[2(\varphi')^2+\parallel W\parallel^2]^2}.
\end{aligned}
$$

Proof. The proof is similar to proof of Theorem 3.1. □

Corollary 3.2. *If α is a helix, then the Frenet apparatus of the $\bar{n}\bar{b}-$Smarandache curve of its Natural mate curve is given by*

$$
T_{\beta_2} = -n, \quad N_{\beta_2} = \frac{1}{C_1}t - \frac{C}{C_1}b, \quad B_{\beta_2} = \frac{C}{C_1}t + \frac{1}{C_1}b,
$$
$$
\kappa_{\beta_2} = \sqrt{2}, \quad \tau_{\beta_2} = 0,
$$

where $\dfrac{\tau}{\kappa} = C =$ constant and $C_1 = \sqrt{1+C^2}$.

Definition 3.4. *Let $(\alpha, \bar{\alpha})$ be a Natural pair in $\mathbb{E}^3$ and $\{\bar{t}, \bar{n}, \bar{b}\}$ be the Frenet frame of the Natural mate curve $\bar{\alpha}$. The $\bar{t}\bar{b}-$Smarandache curve is defined by*

$$
\beta_3(s) = \frac{1}{\sqrt{2}}(\bar{t}(\bar{s}) + \bar{b}(\bar{s})). \tag{3.21}
$$

Theorem 3.3. *The Frenet apparatus of the $\bar{t}\bar{b}-$Smarandache curve is given by*

$$
T_{\beta_3} = \frac{(\varphi'\cos\varphi - \kappa)t + (-\varphi'\sin\varphi + \tau)b}{|\varphi' - \parallel W\parallel|}, \tag{3.22}
$$

$$
N_{\beta_3} = \frac{\lambda_1 t + \parallel W\parallel(\varphi' - \parallel W\parallel)^2 n + \lambda_2 b}{\sqrt{\lambda_1^2 + \parallel W\parallel^2(\varphi' - \parallel W\parallel)^4 + \lambda_2^2}}, \tag{3.23}
$$

$$
B_{\beta_3} = \frac{\sigma_1 t + \sigma_2 n + \sigma_3 b}{(\varphi' - \parallel W\parallel)\sqrt{(\lambda_1^2 + \parallel W\parallel^2(\varphi' - \parallel W\parallel)^4 + \lambda_2^2)}}, \tag{3.24}
$$

$$
\kappa_{\beta_3} = \frac{\sqrt{2(\lambda_1^2 + \parallel W\parallel^2(\varphi' - \parallel W\parallel)^4 + \lambda_2^2)}}{|(\varphi' - \parallel W\parallel)|^3}, \tag{3.25}
$$

$$
\tau_{\beta_3} = \frac{\sqrt{2}(\varphi' - \parallel W\parallel)(\sigma_1\eta_1 + \sigma_2\eta_2 + \sigma_3\eta_3)}{\lambda_1^2 + \parallel W\parallel^2(\varphi' - \parallel W\parallel)^4 + \lambda_2^2}, \tag{3.26}
$$

where $W = \tau t + \kappa b$ and the coefficients are given by

$$
\begin{aligned}
\lambda_1 &= -(\varphi')^3\sin\varphi + \kappa'(\parallel W\parallel - \varphi') + \tau(\varphi')^2 + \parallel W\parallel'(\varphi'\cos\varphi - \kappa),\\
\lambda_2 &= -(\varphi')^3\cos\varphi - \tau'(\parallel W\parallel - \varphi') + \kappa(\varphi')^2 - \parallel W\parallel'(\varphi'\sin\varphi - \tau),
\end{aligned}
$$

$$
\begin{aligned}
\sigma_1 &= -\parallel W\parallel(\varphi' - \parallel W\parallel)^2(-\varphi'\sin\varphi + \tau),\\
\sigma_2 &= \lambda_1(-\varphi'\sin\varphi + \tau) - \lambda_2(\varphi'\cos\varphi - \kappa),\\
\sigma_3 &= \parallel W\parallel(\varphi' - \parallel W\parallel)^2(\varphi'\cos\varphi - \kappa),
\end{aligned}
$$

$$\begin{aligned}
\eta_1 &= \left(\frac{\lambda_1}{(\varphi' - \parallel W \parallel)^3}\right)' - \frac{\kappa \parallel W \parallel}{\varphi' - \parallel W \parallel}, \\
\eta_2 &= \left(\frac{\parallel W \parallel}{\varphi' - \parallel W \parallel}\right)' + \frac{\kappa\lambda_1 - \tau\lambda_2}{(\varphi' - \parallel W \parallel)^3}, \\
\eta_3 &= \left(\frac{\lambda_2}{(\varphi' - \parallel W \parallel)^3}\right)' + \frac{\tau \parallel W \parallel}{\varphi' - \parallel W \parallel}.
\end{aligned}$$

Proof. The proof is similar to proof of Theorem 3.1. □

Corollary 3.3. *If α is a helix, then the Frenet apparatus of the $\bar{t}\bar{b}-$Smarandache curve of its Natural mate curve is given by*

$$T_{\beta_3} = -\frac{1}{C_1}t + \frac{C}{C_1}b, \quad N_{\beta_3} = n, \quad B_{\beta_3} = -\frac{C}{C_1}t - \frac{1}{C_1}b,$$
$$\kappa_{\beta_3} = \sqrt{2}, \quad \tau_{\beta_3} = 0,$$

where $\frac{\tau}{\kappa} = C =$ constant and $C_1 = \sqrt{1 + C^2}$.

Definition 3.5. *Let $(\alpha, \bar{\alpha})$ be a Natural pair in $\mathbb{E}^3$ and $\{\bar{t}, \bar{n}, \bar{b}\}$ be the Frenet frame of the Natural mate curve $\bar{\alpha}$. The $\bar{t}\bar{n}\bar{b}-$Smarandache curve is defined by*

$$\beta_4(s) = \frac{1}{\sqrt{3}}(\bar{t}(\bar{s}) + \bar{n}(\bar{s}) + \bar{b}(\bar{s})). \tag{3.27}$$

Theorem 3.4. *The Frenet apparatus of the $\bar{t}\bar{n}\bar{b}-$Smarandache curve is given by*

$$T_{\beta_4} = \frac{[\varphi'(\cos\varphi + \sin\varphi) - \kappa]t - \parallel W \parallel n + [\varphi'(\cos\varphi - \sin\varphi) + \tau]b}{\sqrt{(\varphi')^2 + \parallel W \parallel^2 + (\varphi' - \parallel W \parallel)^2}}, \tag{3.28}$$

$$N_{\beta_4} = \frac{\lambda_1 t + \lambda_2 n + \lambda_3 b}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}, \tag{3.29}$$

$$B_{\beta_4} = \frac{\sigma_1 t + \sigma_2 n + \sigma_3 b}{\sqrt{(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)[(\varphi')^2 + \parallel W \parallel^2 + (\varphi' - \parallel W \parallel)^2]}}, \tag{3.30}$$

$$\kappa_{\beta_4} = \frac{\sqrt{3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}}{[(\varphi')^2 + \parallel W \parallel^2 + (\varphi' - \parallel W \parallel)^2]^2}, \tag{3.31}$$

$$\tau_{\beta_4} = \frac{\sqrt{3}[(\varphi')^2 + \parallel W \parallel^2 + (\varphi' - \parallel W \parallel)^2](\sigma_1\eta_1 + \sigma_2\eta_2 + \sigma_3\eta_3)}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}, \tag{3.32}$$

where $W = \tau t + \kappa b$ and the coefficients are given by

$$\begin{aligned}
\lambda_1 &= [\cos\varphi(\varphi'' + (\varphi')^2) + \sin\varphi(\varphi'' - (\varphi')^2) - \kappa' + \kappa \parallel W \parallel] \cdot [(\varphi')^2 + \parallel W \parallel^2 + (\varphi' - \parallel W \parallel)^2] \\
&- (\varphi'(\cos\varphi + \sin\varphi) - \kappa)((\varphi')^2 + \parallel W \parallel^2 - \varphi' \parallel W \parallel)', \\
\lambda_2 &= [\parallel W \parallel (\varphi' - \parallel W \parallel) - \parallel W \parallel'] \cdot [(\varphi')^2 + \parallel W \parallel^2 + (\varphi' - \parallel W \parallel)^2] \\
&+ \parallel W \parallel ((\varphi')^2 + \parallel W \parallel^2 - \varphi' \parallel W \parallel)', \\
\lambda_3 &= [\cos\varphi(\varphi'' - (\varphi')^2) - \sin\varphi(\varphi'' + (\varphi')^2) + \tau' - \tau \parallel W \parallel] \cdot [(\varphi')^2 + \parallel W \parallel^2 + (\varphi' - \parallel W \parallel)^2] \\
&- (\varphi'(\cos\varphi - \sin\varphi) + \tau)((\varphi')^2 + \parallel W \parallel^2 - \varphi' \parallel W \parallel)',
\end{aligned}$$

$$\begin{aligned}\sigma_1 &= -\lambda_2[\varphi'(\cos\varphi-\sin\varphi)+\tau]-\lambda_3 \parallel W \parallel,\\ \sigma_2 &= \lambda_1[\varphi'(\cos\varphi-\sin\varphi)+\tau]-\lambda_3[\varphi'(\cos\varphi+\sin\varphi)-\kappa],\\ \sigma_3 &= \lambda_1 \parallel W \parallel +\lambda_2[\varphi'(\cos\varphi+\sin\varphi)-\kappa],\end{aligned}$$

$$\begin{aligned}\eta_1 &= \left(\frac{\lambda_1}{((\varphi')^2+\parallel W\parallel^2+(\varphi'-\parallel W\parallel)^2)^2}\right)'-\frac{\kappa\lambda_2}{((\varphi')^2+\parallel W\parallel^2+(\varphi'-\parallel W\parallel)^2)^2},\\ \eta_2 &= \left(\frac{\lambda_2}{((\varphi')^2+\parallel W\parallel^2+(\varphi'-\parallel W\parallel)^2)^2}\right)'+\frac{\kappa\lambda_1-\tau\lambda_3}{((\varphi')^2+\parallel W\parallel^2+(\varphi'-\parallel W\parallel)^2)^2},\\ \eta_3 &= \left(\frac{\lambda_3}{((\varphi')^2+\parallel W\parallel^2+(\varphi'-\parallel W\parallel)^2)^2}\right)'+\frac{\tau\lambda_2}{((\varphi')^2+\parallel W\parallel^2+(\varphi'-\parallel W\parallel)^2)^2}.\end{aligned}$$

Proof. The proof is similar to proof of Theorem 3.1.

□

Corollary 3.4. *If α is a helix, then the Frenet apparatus of the $\bar{t}\bar{n}\bar{b}-$Smarandache curve of its Natural mate curve is given by*

$$T_{\beta_4}=-\frac{1}{\sqrt{2}C_1}t-\frac{1}{\sqrt{2}}n+\frac{C}{\sqrt{2}C_1}b,\quad N_{\beta_4}=\frac{1}{\sqrt{2}C_1}t-\frac{1}{\sqrt{2}}n-\frac{C}{\sqrt{2}C_1}b,$$
$$B_{\beta_4}=\frac{C}{C_1}t+\frac{1}{C_1}b,\quad \kappa_{\beta_4}=\frac{\sqrt{3}}{\sqrt{2}},\quad \tau_{\beta_4}=0,$$

where $\frac{\tau}{\kappa}=C=$ constant and $C_1=\sqrt{1+C^2}$.

Example 1. *Given the slant helix*

$$\alpha(\bar{s})=\left(\frac{3}{4}\cos(\bar{s})+\frac{\cos(3\bar{s})}{12},\frac{3}{4}\sin(\bar{s})+\frac{\sin(3\bar{s})}{12},-\frac{\sqrt{3}}{2}\cos(\bar{s})\right),\quad \bar{s}\in[0,2\pi].$$

After simple computation, we get

$$t=\left(-\frac{3}{4}\sin(\bar{s})-\frac{\sin(3\bar{s})}{4},\frac{3}{4}\cos(\bar{s})+\frac{\cos(3\bar{s})}{4},\frac{\sqrt{3}}{2}\sin(\bar{s})\right),$$
$$n=\left(-\frac{\sqrt{3}}{2}\cos(2\bar{s}),-\frac{\sqrt{3}}{2}\sin(2\bar{s}),\frac{1}{2}\right),$$
$$b=\left(\frac{1}{4}(3\cos(\bar{s})-\cos(3\bar{s})),\sin^3(\bar{s}),\frac{\sqrt{3}}{2}\cos(\bar{s})\right),$$
$$\kappa=\sqrt{3}\cos(\bar{s}),\quad \tau=\sqrt{3}\sin(\bar{s}).$$

The Natural mate curve of α is the helix

$$\bar{\alpha}(\bar{s})=\left(-\frac{\sqrt{3}}{4}\sin(2\bar{s}),\frac{\sqrt{3}}{4}\cos(2\bar{s}),\frac{\bar{s}}{2}\right),\quad \bar{s}\in[0,2\pi].$$

From equation (2.5), we get $\varphi(\bar{s})=\bar{s}+\varphi_0$. If we choose $\varphi_0=0$, we have that

$$\bar{t} = \Big(-\frac{\sqrt{3}}{2}\cos(2\bar{s}), -\frac{\sqrt{3}}{2}\sin(2\bar{s}), \frac{1}{2}\Big),$$

$$\bar{n} = \Big(\sin(2\bar{s}), -\cos(2\bar{s}), 0\Big),$$

$$\bar{b} = \Big(\frac{1}{2}\cos(2\bar{s}), \frac{1}{2}\sin(2\bar{s}), \frac{\sqrt{3}}{2}\Big),$$

$$\bar{\kappa} = \sqrt{3}, \quad \bar{\tau} = 1.$$

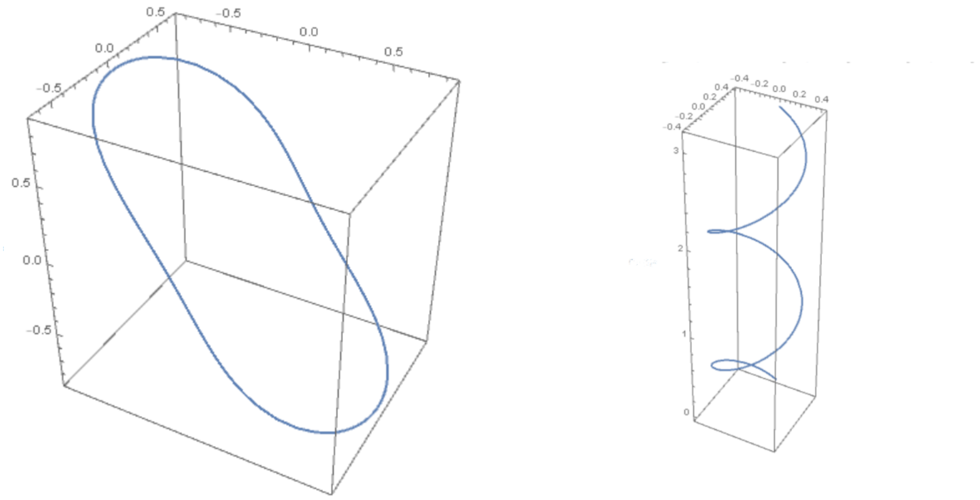
The $\bar{t}\bar{n}, \bar{n}\bar{b}, \bar{t}\bar{b}, \bar{t}\bar{n}\bar{b}-$*Smarandache curves are, respectively*

$$\beta_1(s) = \frac{1}{\sqrt{2}}\Big(-\frac{\sqrt{3}}{2}\cos\Big(\frac{2\sqrt{2}}{\sqrt{7}}s\Big) + \sin\Big(\frac{2\sqrt{2}}{\sqrt{7}}s\Big), -\frac{\sqrt{3}}{2}\sin\Big(\frac{2\sqrt{2}}{\sqrt{7}}s\Big) - \cos\Big(\frac{2\sqrt{2}}{\sqrt{7}}s\Big), \frac{1}{2}\Big),$$

$$\beta_2(s) = \frac{1}{\sqrt{2}}\Big(\frac{1}{2}\cos\Big(\frac{2\sqrt{2}}{\sqrt{5}}s\Big) + \sin\Big(\frac{2\sqrt{2}}{\sqrt{5}}s\Big), \frac{1}{2}\sin\Big(\frac{2\sqrt{2}}{\sqrt{5}}s\Big) - \cos\Big(\frac{2\sqrt{2}}{\sqrt{5}}s\Big), \frac{\sqrt{3}}{2}\Big),$$

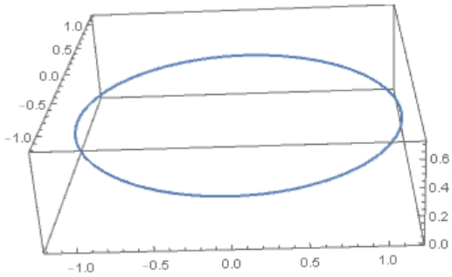
$$\beta_3(s) = \frac{1}{\sqrt{2}}\Big(\frac{\sqrt{3}-1}{2}\cos\Big(\frac{2\sqrt{2}}{\sqrt{3}-1}s\Big), -\frac{\sqrt{3}-1}{2}\sin\Big(\frac{2\sqrt{2}}{\sqrt{3}-1}s\Big), \frac{\sqrt{3}+1}{2}\Big),$$

$$\begin{aligned}\beta_4(s) = \frac{1}{\sqrt{3}}\Big(&\frac{1-\sqrt{3}}{2}\cos\Big(\frac{2\sqrt{3}}{\sqrt{8-2\sqrt{3}}}s\Big) + \sin\Big(\frac{2\sqrt{3}}{\sqrt{8-2\sqrt{3}}}s\Big),\\ &\frac{1-\sqrt{3}}{2}\sin\Big(\frac{2\sqrt{3}}{\sqrt{8-2\sqrt{3}}}s\Big) - \cos\Big(\frac{2\sqrt{3}}{\sqrt{8-2\sqrt{3}}}s\Big), \frac{1+\sqrt{3}}{2}\Big).\end{aligned}$$

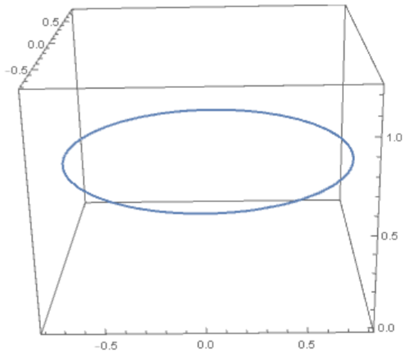


(a) Natural curve α (b) Natural mate curve $\bar{\alpha}$

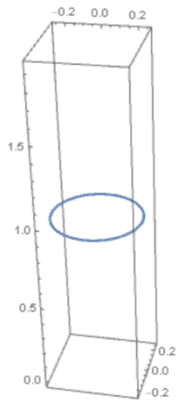
Fig. 1. Natural Curves Pair $(\alpha, \bar{\alpha})$



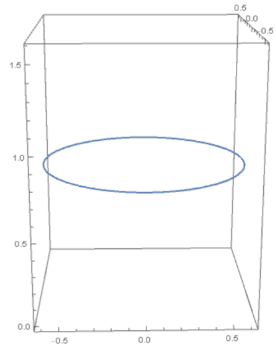
(a) $\bar{t}\bar{n}-$Smarandache curve



(b) $\bar{n}\bar{b}-$Smarandache curve



(c) $\bar{t}\bar{b}-$Smarandache curve



(d) $\bar{t}\bar{n}\bar{b}-$Smarandache curve

Fig. 2. Samarandache's curves of $\bar{\alpha}$

4 CONCLUSIONS

Smarandache curves have been receiving increasing attention from researchers due to their utility in differential geometry and their applications in various fields such as physics, engineering, robotics, computer graphics, and others. In this work, we study a special class of curves called Smarandache curves of curves, which are the natural mate of a curve α, according to the Frenet frame in three-dimensional Euclidean space. We calculate the Frenet apparatus for these curves and provide as a corollary the Frenet apparatus for the case where the curve α is a helix. Additionally, we present an example in which we calculate the Smarandache curves of the natural mate for the case where α is a slant helix. We hope that the results presented here contribute to the study of these curves and to the development of their applications.

DISCLAIMER (ARTIFICIAL INTELLIGENCE)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models

(ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

ACKNOWLEDGEMENT

We would like to thank the referees for their valuable comments and suggestions.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

[1] Matsuda H, Yorozu S., Notes on Bertrand curves. Yokohama Mathematical Journal. 2001;47(1):31-42.

[2] $\ddot{O}$zdamar E, Kazaz M. Mannheim partner curves in 3-dimensional Euclidean space, International Journal of Physical Sciences. 2008;3(3):139-144.

[3] Litvin FL, Fuentes A. Gear geometry and applied theory. Cambridge University Press; 2004.

[4] Turhan T. Magnetic trajectories in three-dimensional Lie groups. Mathematical Methods in the Applied Sciences. 2020;43(5):2747-2758.

[5] Ceylan AY, Turhan T, Tukel GO. On the geometry of rational Bézier curves. Honam Mathematical Journal. 2021;43(1):88-99.

[6] $\ddot{O}$zkan TG, Şahin B, Turhan T. Certain curves along Riemannian submersions. Filomat. 2023;37(3):905-913.

[7] Şenyurt S, Çalişkan A. Curves and ruled surfaces according to alternative frame in dual space, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 2020;69(1):684-698.

[8] Gok I, Uzunoglu B, Yayli Y. A new approroach on curves of constant precession. Applied Mathematics and Computation. 2016;275(1):317-323.

[9] Choi J, Kim Y. Associated curves of a Frenet curve and their Applications. Applied Mathematics and Computation. 2012;218:9116-9124.

[10] Abdel-Baky RA, Naghi MF. Sweeping surfaces with Natural mate curve of a spatial curve in Euclidean 3-space, WSEAS Transactions on Mathematics. 2020;19:581-588.

[11] Deshmukh, S., Chen, B.Y. and Alghanemi, A. Natural mates of frenet curves in Euclidean 3-space. Turkish Journal of Mathematics. 2018;42:2826-2840.

[12] Camci Ç, Chen BY, Ilarslan K, Uçum A. Sequential natural mates of Frenet curves in Euclidean 3-space, Journal of Geometry. 2021;112:46.

[13] Mak M, Natural and conjugate mates of frenet curves in three-dimensinal lie group. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics. 2021;70(1):522-540.

[14] Ashbacher C. Smarandache geometries. Smarandache Notions Journal. 1997;8(13)212-215.

[15] do Carmo MP. Differential geometry of curves and surfaces. Prentice-Hall, Englewood Cliffs, NJ; 1976.

[16] Rabouski D, Smarandache F, Borisova L. Neutrosophic methods in general relativity. Phoenix: Hexis Publishers; 2005.

[17] Savas M, Yakut AT, Tamirci T. The smarandache curves on $\mathbb{H}_0^2$, Gazi University Journal of Science. 2016;29(1):69-77.

[18] Ali AT. Smarandache Curves in the Euclidean Space. International Journal of Mathematical Combinatorics. 2010;2:30-36.

[19] Şenyurt S, Çalişkan A, Çelik, $\ddot{U}$. Smarandache curves of bertrand curves pair according to frenet frame. Boletim da Sociedade Paranaense de Matemática. 2021;39(5):163-173.

[20] Şenyurt S, Kaya G. NC-Smarandache Curve and NW-Smarandache Curve According to Alternative Frame, Proceedings of International Conference on Mathematics and Mathematics Education (ICMME 2018) Turk. J. Math. Comput. Sci. 2018;10:269-274.

[21] Solouma EM, Wageeda MM. Special smarandache curves according to bishop frame in euclidean spacetime. International Journal of Mathematical Combinatorics. 2017;1:01-09.

[22] Taşk$\ddot{o}$pr$\ddot{u}$ K, Tosun M. Smarandache curves according to sabban frame on $\mathbb{S}^2$. Boletim da Sociedade Paranaense de Matemática. 2014;32(1):51-59.

[23] Çalışkan A., Şenyurt S. Smarandache curves in terms of sabban frame of fixed pole curve. Boletim da Sociedade Paranaense de Matemática. 2016;34(2):53-62.

[24] Çalışkan A, Şenyurt S. N^*C^*- Smarandache Curves of Mannheim Curve Couple According to Frenet Frame. International J.Math. Combin. 2015;1:1-13.

[25] Cetin M, Tuncer Y, Karacan MK. Smarandache curves according to bishop frame in Euclidean 3-Space. Gen. Math. Notes. 2014;20(2):50-66.

[26] Nurkan SK, G$\ddot{u}$ven, $\dot{I}$ A, Karakan MK. Characterizations of adjoint curves in euclidean 3-space, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences. 2018;89:155-161.

[27] Turgut M, Yilmaz S. Smarandache curves in minkowski space-time. International Journal of Mathematical Combinatorics. 2008;351-55.

Peer-review history:
The peer review history for this paper can be accessed here:
https://www.sdiarticle5.com/review-history/121767