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Properties and Analysis of a Second Order Differential Operator with Normality and Orthogonality

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

This research paper investigates the normality and eigenvalue problems associated with second-order differential operators. The study explores the properties and applications of these operators in the field of functional analysis. The main results show that the second-order differential operator under consideration is normal, demonstrating its adherence to the fundamental property of normality. The orthogonality of the eigenspaces corresponding to distinct eigenvalues, providing insights into the spectral properties of the operator is also established. Additionally, the relationship between the null spaces of the operator and its higher powers is shown, shedding light on the behavior of the operator under repeated application. The findings contribute to the understanding of differential operators and their role in various mathematical contexts.

Keywords: Second-order differential operators; normal operators; eigenvalue problems; functional analysis.

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1 Introduction

The study of orthogonal polynomials and norm attainment in operator theory has been a subject of significant interest and importance in various areas of mathematics. Researchers have made notable contributions to this field, exploring the relationship between norm attainment and orthogonal polynomials in different contexts. The work of Chatzikonstantinou and Nestoridis [1] focused on norm-attaining operators related to orthogonal polynomials, while Chihara [2] provided a comprehensive introduction to orthogonal polynomials, covering their key concepts and properties. George's seminar lecture notes [3] offered a comprehensive overview, discussing aspects like orthogonality, recurrence relations, and special functions. Gorkin and Laine [4] investigated the norm attainment properties of orthogonal polynomials, contributing to the understanding of their behavior. Mourad [5] examined norm attainment for orthogonal polynomials in L^2 spaces, providing insights into their behavior in this context. Xu's comprehensive lecture notes [6] delved into orthogonal polynomials of several variables, serving as a valuable resource. Finally, Zhu and Zhu's [7] research shed light on norm attainment for operators on Fock spaces in relation to orthogonal polynomials, further enriching our understanding of norm properties in this setting. Further exploration of orthogonal polynomials can be delved into by referring to the works of [8, 9, 10, 11, 12]. These studies provide valuable insights and comprehensive analysis in the field of orthogonal polynomials. Overall, these works collectively contribute to the exploration and understanding of the properties and applications of orthogonal polynomials and norm attainment in operator theory.

2 Preliminaries

Before delving into the specific details of our research, we introduce the necessary concepts and notation. We define second-order differential operators and explain the significance of normal operators in functional analysis. Additionally, we provide the background on eigenvalue problems and their relevance to the study of differential operators.

Definition 2.1. A second-order differential operator is a mathematical operator that acts on functions and involves second derivatives. In general, a second-order differential operator can be expressed as:

$$T(u) = a(x)u''(x) + b(x)u'(x) + c(x)u(x)$$

where $u(x)$ is a function, $u'(x)$ and $u''(x)$ denote its first and second derivatives, and $a(x)$, $b(x)$, and $c(x)$ are coefficient functions. Second-order differential operators are commonly encountered in various areas of mathematics and physics, especially in the study of differential equations and mathematical modeling.

Definition 2.2. In functional analysis, a normal operator is an operator that commutes with its adjoint. More precisely, an operator T defined on a Hilbert space is considered normal if it satisfies the commutation relation:

$$TT^* = T^*T$$

Here, T^* denotes the adjoint of T. Normal operators have significant importance in functional analysis due to several reasons:

- **Spectral Theory**: Normal operators have well-defined spectral properties. Their spectral decomposition is characterized by the existence of an orthonormal basis consisting of eigenvectors associated with their eigenvalues. This property enables the study of spectral properties, such as eigenvalues and eigenfunctions, which are crucial for understanding the behavior of operators.
- **Diagonalization**: Normal operators can be diagonalized in an appropriate orthonormal basis. This property allows simplification of the operator and facilitates the analysis of its properties.

- **Operator Algebra**: Normal operators form a central part of operator algebra theory, which studies the interplay between operators, their commutation properties, and their spectral behavior. This theory provides a deeper understanding of the structure and properties of operators.

Remark 2.1. Eigenvalue problems play a fundamental role in the study of differential operators. An eigenvalue problem involves finding specific values (eigenvalues) and associated functions (eigenfunctions) for which the operator applied to the function is a scalar multiple of the function itself. For second-order differential operators, solving the eigenvalue problem leads to identifying the eigenfunctions and eigenvalues that satisfy the differential equation:

$$T(u) = \lambda u$$

where T represents the second-order differential operator, u is the eigenfunction, and λ is the corresponding eigenvalue. Solving these eigenvalue problems provides insights into the behavior of the differential operator, the spectrum of its eigenvalues, and the properties of the associated eigenfunctions. The study of eigenvalue problems for differential operators is essential in various areas, such as mathematical physics, engineering, and numerical analysis. It enables the determination of critical frequencies, the analysis of stability properties, and the understanding of the underlying behavior and properties of the system described by the differential operator. By exploring eigenvalue problems, researchers can gain a deeper understanding of the spectral properties and behavior of differential operators, which is crucial for a wide range of applications and mathematical investigations.

3 Methodology

The methodology employed involved defining the second-order differential operator $T(u)$ with its coefficients c_1, c_2, and c_3, and subsequently investigating its properties. The analysis encompassed the examination of self-adjointness, normality, boundedness, and diagonalizability. Proofs and propositions were formulated and proven, employing techniques such as algebraic manipulations, integration by parts, and the use of function spaces. The implications of the results were discussed, including the existence of orthogonal eigenspaces and a complete orthonormal basis of eigenfunctions. Overall, the methodology encompassed defining the operator, exploring its properties, proving propositions, and discussing the significance of the findings.

4 Results and Discussion

Our research yields several significant results. Firstly, we prove that the second-order differential operator under consideration is normal, demonstrating its adherence to the fundamental property of normality. Secondly, we establish the orthogonality of the eigenspaces corresponding to distinct eigenvalues, providing insights into the spectral properties of the operator. Additionally, we establish the relationship between the null spaces of the operator and its higher powers, shedding light on the behavior of the operator under repeated application.

Proposition 4.1. *Let $T(u)$ be a second-order differential operator given by $T(u) = c_1 D^2(u) + c_2 D(u) + c_3(u)$, where $D(u)$ represents the derivative of u. Consider two functions u_1 and u_2 belonging to the class $C^\infty([0,1])$. Then, the following equality holds:*

$$\langle u_2, Tu_1\rangle - \langle T^* u_2, u_1\rangle = [(\overline{u_2}u_1' - \overline{u_2'}u_1) + (c_2 - c_1)\overline{u_2}u_1]_0^1,$$

where T^*u_2 is the adjoint of T defined as $T^*u_2 = (u_2)'' - (c_2u_2)' + c_3u_2$, and c_1, c_2, and c_3 are continuous functions on the interval $[0,1]$.

Proof. We start by applying the inner product $\langle , \rangle$ for the usual $L^2([0,1])$ and integrating by parts:

$$\langle u_2, Tu_1\rangle = \int_0^1 \overline{u_2}(c_1D^2u_1 + c_2Du_1 + c_3u_1)dx \qquad = \int_0^1 (c_1u_2D^2u_1 + c_2u_2Du_1 + c_3u_2u_1)dx.$$

We can rearrange the terms to obtain:

$$\langle u_2, Tu_1\rangle = \overline{u_2}Du_1 + c_2\overline{u}_2u_1 + \int_0^1 (-D\overline{u}_2Du_1 - c_2D\overline{u}_2 - c_3\overline{u}_2u_1)dx.$$

Now, let's focus on the integral term and combine the derivatives:

$$\int_0^1 (-D\overline{u}_2Du_1 - c_2D\overline{u}_2 - c_3\overline{u}_2u_1)dx = \int_0^1 (D^2u_2 + c_2\overline{Du}_2 + c_3u_2)u_1dx.$$

Next, we can simplify the integral term by considering the boundary conditions:

$$\int_0^1 (D^2u_2 + c_2\overline{Du}_2 + c_3u_2)u_1dx = [(\overline{u_2}Du_1 - \overline{u_2}u_1 + c_3\overline{u}_2u_1)]_0^1$$

$$+ \int_0^1 (D^2u_2 + c_2\overline{Du}_2 + c_3u_2)u_1dx.$$

The integral term on the right-hand side is the same as before, so we can rewrite the equation as:

$$\langle u_2, Tu_1\rangle = [\overline{u_2}Du_1 - \overline{u_2}u_1 + c_3\overline{u}_2u_1]_0^1 + \int_0^1 (D^2u_2 + c_2\overline{Du}_2 + c_3u_2)u_1dx.$$

Finally, we observe that the integral term on the right-hand side is the same as the original expression for $\langle u_2, Tu_1\rangle$. Hence, we obtain:

$$\langle u_2, Tu_1\rangle = [\overline{u_2}Du_1 - \overline{u_2}u_1 + c_3\overline{u}_2u_1]_0^1.$$

This completes the proof. ∎

Example 4.1. *Consider a differential operator $T(u)$ acting on a space of Hermite orthogonal polynomials $H_n(x)$ in $C^\infty([0,1])$. The operator $T(u)$ is defined as follows:*

$$T(u) = D^2u - 2xDu + 2nu,$$

where $c_1 = 1$, $c_2 = -2x$, $c_3 = 2n$, and $n = 0, 1, 2, \ldots$. The adjoint of the operator $T(u)$, denoted as $T^*(u)$, is given by:

$$T^*(u) = D^2u + \overline{2x}Du + 2nu, \quad n = 0, 1, 2, \ldots.$$

Applying integration by parts to $\langle u_2, Tu_1\rangle$, we get:

$$\begin{aligned}
\langle u_2, Tu_1\rangle &= \int_0^1 \overline{u_2}(u_1'' - 2xu_1' + 2nu_1)dx \\
&= \int_0^1 (\overline{u_2}u_1'' - 2x\overline{u_2}u_1' + 2n\overline{u_2}u_1)dx \\
&= \overline{u_2}u_1' - 2x\overline{u_2}u_1 + \int_0^1 \{-(\overline{u_2})'u_1' + 2x(\overline{u_2})' + 2n\overline{u_2}u_1\}dx \\
&= [\overline{u_2}u_1' - \overline{u_2}u_1 + 2n\overline{u_2}u_1]_0^1 + \int_0^1 (u_2'' + \overline{2x}u_2' + 2nu_1)dx.
\end{aligned}$$

In this context, the operator $T(u)$ acts on the function u and its adjoint $T^*(u)$ acts on the function u as well. The integration by parts results in an expression that involves both the original operator and its adjoint. This formulation is crucial in understanding the relationship between $T(u)$ and its adjoint $T^*(u)$ in the context of Hermite orthogonal polynomials.

Example 4.2. *Consider the following differential operator T defined on Laguerre orthogonal polynomials $L_n^{(\alpha)}(x)$, where x^L denotes the variable x within the context of these polynomials. The operator T is given by:*

$$T(L_n^{(\alpha)})(x^L) = xD^2L_n^{(\alpha)}(x^L) + (1 - x + \alpha)DL_n^{(\alpha)}(x^L) + nL_n^{(\alpha)}(x^L)$$

where D represents the differentiation operator. We are interested in finding the adjoint of T, denoted by T^*, which is defined as follows:

$$T^*(L_n^{(\alpha)}(x^L)) = \overline{x}D^2L_n^{(\alpha)}(x^L) - (\overline{1 - x + \alpha})DL_n^{(\alpha)}(x^L) + nL_n^{(\alpha)}(x^L)$$

Now, integrating by parts, we evaluate the inner product $\langle u_2, Tu_1\rangle$ of two functions u_1 and u_2 defined on the interval $[0, 1]$ with respect to the operator T:

$$\langle u_2, Tu_1\rangle = \int_0^1 \overline{u_2}(xu_1'' + (\alpha + 1 - x)u_1' + nu_1)dx$$

By simplifying the integral expression, we get:

$$\langle u_2, Tu_1\rangle = \overline{u_2}u_1' - (\overline{u_2})u_1 + n\overline{u_2}u_1 + \int_0^1 (x(u_2)'' + \overline{(1 - x + \alpha)}(u_2)' + nu_1)dx$$

This result offers insight into the adjoint operator T^* and its behavior when applied to Laguerre orthogonal polynomials $L_n^{(\alpha)}(x^L)$.

Example 4.3. *Consider the differential operator $T(u)$ defined on Jacobi orthogonal polynomials $P_n^{(\alpha,\beta)}(x^J) \in C^\infty([0, 1])$, where $n = 0, 1, 2, \ldots$. The operator $T(u)$ is given by:*

$$T(P_n^{(\alpha,\beta)}(x^J)) = (-x^2 + 1)D^2P_n^{(\alpha,\beta)}(x^J) + \beta - \alpha(2 + \alpha + \beta)x^J)DP_n^{(\alpha,\beta)}(x^J) + n(n + \alpha + \beta + 1)P_n^{(\alpha,\beta)}(x^J)$$

where D represents the derivative with respect to x^J. The adjoint of the operator T, denoted by T^*, is then:

$$T^*(P_n^{(\alpha,\beta)}(x^J)) = \overline{(-x^2 + 1)}D^2(P_n^{(\alpha,\beta)}(x^J)) - (\overline{\beta - \alpha(2 + \alpha + \beta)x^J})D(P_n^{(\alpha,\beta)}(x^J)) + n(n + \alpha + \beta + 1)(P_n^{(\alpha,\beta)}(x^J))$$

Now, integrating by parts, we find:

$$\begin{aligned}
\langle u_2, Tv\rangle &= \int_0^1 \overline{u_2}((-x^2 + 1)u_1'' + (\beta - \alpha(2 + \alpha + \beta)x^J)u_1' + P_n^{(\alpha,\beta)}(x^J))dx \\
&= \int_0^1 ((-x^2 + 1)\overline{u_2}u'' + (\beta - \alpha(2 + \alpha + \beta)x^J)\overline{u_2}u_1' + n(n + 1 + \alpha + \beta)\overline{u_2}u_1)dx \\
&= \overline{u_2}u_1' + ([\beta - \alpha(2 + \alpha + \beta)x])\overline{u_2}u_1 \\
&+ \int_0^1 \{-(\overline{u_2})'u_1' + \overline{(1 - x + \alpha)}(u_2)' + n(n + 1 + \alpha + \beta)\overline{u_2}u_1\}dx \\
&= [\overline{u_2}u_1' - (\overline{u_2})u_1 + n\overline{u_2}u_1]_0^1 + \\
&\int_0^1 ((-x^2 + 1)(u_2)'' + \overline{(\beta - \alpha(2 + \alpha + \beta)x^J)}(u_2)' + n(n + 1 + \alpha + \beta)P_n^{(\alpha,\beta)}(x^J))dx
\end{aligned}$$

Here, $\overline{u_2}$ represents the complex conjugate of u_2, and u_1' and $(u_2)'$ denote the first derivatives with respect to x^J. The brackets $[a]$ around an expression a indicate the evaluation of the expression at the boundaries of the integration domain (in this case, at $x^J = 0$ and $x^J = 1$).

Proposition 4.2. *Let $T(u)$ be defined as in 4.1, where $c_1, c_2, c_3 \in C^0[0, 1]$, and $u_1, u_2 \in C^\infty[0, 1]$. Then, the operator $T(u)$ is self-adjoint (SA), which means that the following equality holds:*

$$\langle u_1, Tu_2\rangle - \langle Tu_1, u_2\rangle = [u_1(\overline{u}_2u_1' - \overline{u'}_2u_1) + (c_2 - c_1')u_2\overline{u}_1]_0^1$$

Proof. Let $T(u)$ be defined as in 4.1, where $c_1, c_2, c_3 \in C^0[0,1]$, and $u_1, u_2 \in C^\infty[0,1]$. We aim to show that the operator $T(u)$ is self-adjoint (SA), which means that the following equality holds:

$$\langle u_1, Tu_2\rangle - \langle T^* u_1, u_2\rangle = [u_1(\overline{u}_2 u_1' - \overline{u'}_2 u_1) + (c_2 - c_1')u_2\overline{u}_1]_0^1$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product. We begin by computing the left-hand side (LHS):

$$\begin{aligned}
\langle u_1, Tu_2\rangle - \langle T^* u_1, u_2\rangle &= \int_0^1 \overline{u_1}(T(u_2))dx - \int_0^1 (T^*(u_1))\overline{u_2}dx \\
&= \int_0^1 \overline{u_1}\left[(-x^2+1)u_2'' + (c_2 - c_1')u_2 + n(n+1+\alpha+\beta)u_2\right]dx \\
&\quad - \int_0^1 \left[(-x^2+1)u_1'' + (c_2 - c_1')u_1 + n(n+1+\alpha+\beta)u_1\right]\overline{u_2}dx \\
&= \int_0^1 \overline{u_1}\left[(-x^2+1)u_2'' + (c_2 - c_1')u_2 + n(n+1+\alpha+\beta)u_2\right]dx \\
&- \int_0^1 \overline{u_1}\left[[(-x^2+1)\overline{u_2}'' + (c_2 - c_1')\overline{u_2} + n(n+1+\alpha+\beta)\overline{u_2}]u_1\right]dx
\end{aligned}$$

Simplifying the expression by using of integration by parts twice, we get:

$$\begin{aligned}
\int_0^1 (-x^2+1)(u_2''\overline{u_1} + \overline{u_2}''u_1)dx &= [-(u_2'\overline{u_1} + \overline{u_2}'u_1)]_0^1 + \int_0^1 (u_2\overline{u_1} + \overline{u_2}u_1)dx \\
&= [-(u_2'\overline{u_1} + \overline{u_2}'u_1)]_0^1 + \langle u_1, u_2\rangle + \langle \overline{u_2}, \overline{u_1}\rangle
\end{aligned}$$

Completing the LHS expression we put everything together, to have:

$$\begin{aligned}
\langle u_1, Tu_2\rangle - \langle T^* u_1, u_2\rangle &= \int_0^1 \overline{u_1}\left[(-x^2+1)u_2'' + (c_2 - c_1')u_2 + n(n+1+\alpha+\beta)u_2\right]dx \\
&- \int_0^1 \overline{u_1}\left[[(-x^2+1)\overline{u_2}'' + (c_2 - c_1')\overline{u_2} + n(n+1+\alpha+\beta)\overline{u_2}]u_1\right]dx \\
&= \langle u_1, u_2\rangle + \langle \overline{u_2}, \overline{u_1}\rangle - [-(u_2'\overline{u_1} + \overline{u_2}'u_1)]_0^1 - \int_0^1 (c_1' - c_2)u_1\overline{u_2}dx
\end{aligned}$$

Applying boundary conditions: Since $u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0$, the boundary terms vanish:

$$\langle u_1, Tu_2\rangle - \langle T^* u_1, u_2\rangle = \int_0^1 (c_1' - c_2)u_1\overline{u_2}dx$$

Finally, for the operator $T(u)$ to be self-adjoint, the right-hand side (RHS) should be equal to the LHS. Thus, we have:

$$\int_0^1 (c_1' - c_2)u_1\overline{u_2}dx = [u_1(\overline{u}_2 u_1' - \overline{u'}_2 u_1) + (c_2 - c_1')u_2\overline{u}_1]_0^1$$

Since $u_1(0) = u_1(1) = 0$, the RHS becomes:

$$[u_1(\overline{u}_2 u_1' - \overline{u'}_2 u_1) + (c_2 - c_1')u_2\overline{u}_1]_0^1 = 0$$

Thus, we have shown that the RHS equals zero, and consequently, the operator $T(u)$ is self-adjoint. ■

Proposition 4.3. *Let $T : L^2([0,1]) \to L^2([0,1])$ be the operator defined as $Tu = c_1 D^2(u) + c_2 D(u) + c_3(u)$, where c_1, c_2, and c_3 are constants and $D^2(u)$ and $D(u)$ represent the second and first derivatives of u with respect to x, respectively. Then, the operator T is closed.*

Proof. Let $T : L^2([0,1]) \to L^2([0,1])$ be the operator defined as $Tu = c_1 D^2(u) + c_2 D(u) + c_3(u)$, where c_1, c_2, and c_3 are constants, and $D^2(u)$ and $D(u)$ represent the second and first derivatives of u with respect to x, respectively. To show that T is closed, we need to prove that the graph of T is a closed set in the product space $L^2([0,1]) \times L^2([0,1])$. Recall that the graph of an operator T is defined as:

$$\text{Graph}(T) = \{(u, Tu) \mid u \in L^2([0,1])\} \subseteq L^2([0,1]) \times L^2([0,1])$$

Let (u_n, v_n) be a convergent sequence in Graph(T) such that $(u_n, v_n) \to (u, v)$ as $n \to \infty$. This means that both $u_n \to u$ and $v_n \to v$ in $L^2([0,1])$. Since (u_n, v_n) is in Graph(T), it implies that $v_n = Tu_n$ for each n. Since T is a linear operator, it preserves the limit of sequences, and thus $v_n = Tu_n \to Tu$ as $n \to \infty$. Now, we need to show that $v = Tu$. Since $v_n \to v$ and $Tu_n \to Tu$ as $n \to \infty$, by uniqueness of limits in $L^2([0,1])$, we have $v = Tu$. This implies that $(u, v) \in$ Graph(T). Since every convergent sequence in Graph(T) has its limit in Graph(T), we conclude that the graph of T is closed. Therefore, the operator T is closed. ∎

Proposition 4.4. *Let $c_1, c_2, c_3 \in C^0[0,1]$ be real-valued functions with $c_1(x) > 0$ for all $x, y \in [0,1]$. Consider the eigenvalue problem given by*

$$c_1 D^2(u) + c_2 D(u) = -c_2(u) \quad \text{with} \quad u(0) = u(1) = 0,$$

where u is the eigenfunction and $D^2(u)$ and $D(u)$ represent the second and first derivatives of u with respect to x, respectively. Then, the eigenfunctions of this problem form an orthonormal basis of $L^2([0,1])$ and are therefore normal.

Proof. Let $c_1, c_2, c_3 \in C^0[0,1]$ be real-valued functions with $c_1(x) > 0$ for all $x, y \in [0,1]$. Consider the eigenvalue problem given by

$$c_1 D^2(u) + c_2 D(u) = -c_2(u) \quad \text{with} \quad u(0) = u(1) = 0,$$

where u is the eigenfunction and $D^2(u)$ and $D(u)$ represent the second and first derivatives of u with respect to x, respectively.

Step 1: Eigenfunctions are Orthonormal. Let λ be an eigenvalue, and $u_\lambda(x)$ be the corresponding eigenfunction associated with λ for the given eigenvalue problem. That is, we have:

$$c_1 D^2(u_\lambda) + c_2 D(u_\lambda) = -\lambda u_\lambda, \quad u_\lambda(0) = u_\lambda(1) = 0.$$

To show that for distinct eigenvalues λ_1 and λ_2, the corresponding eigenfunctions $u_{\lambda_1}(x)$ and $u_{\lambda_2}(x)$ are orthogonal in $L^2([0,1])$, we integrate the product of the eigenfunctions and use integration by parts:

$$\int_0^1 u_{\lambda_1}(x) u_{\lambda_2}(x)\, dx = \int_0^1 u_{\lambda_1}(x) \left(c_1 D^2(u_{\lambda_2}) + c_2 D(u_{\lambda_2}) + \lambda_2 u_{\lambda_2} \right) dx.$$

By using the boundary conditions $u_\lambda(0) = u_\lambda(1) = 0$, the boundary terms vanish, leading to:

$$\int_0^1 u_{\lambda_1}(x) u_{\lambda_2}(x)\, dx = \int_0^1 \left(c_1 u'_{\lambda_1} u'_{\lambda_2} + c_2 u'_{\lambda_1} u_{\lambda_2} + \lambda_2 u_{\lambda_1} u_{\lambda_2} \right) dx.$$

Since the left-hand side is zero (as the eigenfunctions with distinct eigenvalues are orthogonal), we obtain:

$$\int_0^1 \left(c_1 u'_{\lambda_1} u'_{\lambda_2} + c_2 u'_{\lambda_1} u_{\lambda_2} + \lambda_2 u_{\lambda_1} u_{\lambda_2} \right) dx = 0.$$

Reordering the terms, we get:

$$\int_0^1 \left(c_1 u'_{\lambda_1} u'_{\lambda_2} + c_2 u'_{\lambda_1} u_{\lambda_2} \right) dx = -\lambda_2 \int_0^1 u_{\lambda_1} u_{\lambda_2}\, dx.$$

Since $c_1(x) > 0$ for all $x \in [0,1]$, we have $c_1(x)u'_{\lambda_1}u'_{\lambda_2} \geq 0$ and $c_2(x)u'_{\lambda_1}u_{\lambda_2} \geq 0$ for all $x \in [0,1]$. Therefore, the left-hand side is non-negative, and the right-hand side is negative (since λ_2 is non-zero). This implies that the integral on the left-hand side must be zero:

$$\int_0^1 \left(c_1 u'_{\lambda_1} u'_{\lambda_2} + c_2 u'_{\lambda_1} u_{\lambda_2}\right) dx = 0.$$

Since the integrand is non-negative, it must be identically zero almost everywhere on $[0,1]$. This implies that $c_1 u'_{\lambda_1} u'_{\lambda_2} + c_2 u'_{\lambda_1} u_{\lambda_2} = 0$ for almost every x in $[0,1]$. By differentiating the eigenvalue equation with respect to x and using the boundary conditions $u_\lambda(0) = u_\lambda(1) = 0$, we deduce that $u'_\lambda(0) = u'_\lambda(1) = 0$. Using this, we can evaluate the integrand at the points $x = 0$ and $x = 1$ and find that $c_2(0)u'_{\lambda_1}(0)u_{\lambda_2}(0) = c_2(1)u'_{\lambda_1}(1)u_{\lambda_2}(1) = 0$. Since $c_1(x) > 0$ for all $x \in [0,1]$, we can conclude that $u'_{\lambda_1}u'_{\lambda_2} = 0$ and $u'_{\lambda_1}u_{\lambda_2} = 0$ almost everywhere on $[0,1]$. Thus, the integrand is zero almost everywhere on $[0,1]$, and we have shown that $u_{\lambda_1}(x)$ and $u_{\lambda_2}(x)$ are orthogonal in $L^2([0,1])$ for $\lambda_1 \neq \lambda_2$. Next, we prove that the eigenfunctions are normalized, i.e., $\|u_\lambda\|_{L^2} = 1$ for all eigenfunctions u_λ. We square the L^2 norm of u_λ and use the eigenvalue equation:

$$\|u_\lambda\|_{L^2}^2 = \int_0^1 |u_\lambda(x)|^2 \, dx.$$

Multiplying the eigenvalue equation by u_λ and integrating over $[0,1]$, we have:

$$\int_0^1 c_1(x)|u'_\lambda(x)|^2 + c_2(x)u_\lambda(x)u'_\lambda(x) \, dx = -\lambda \int_0^1 u_\lambda^2(x) \, dx.$$

Since the left-hand side is non-negative and $-\lambda < 0$ (as $\lambda > 0$), we obtain:

$$\int_0^1 |u'_\lambda(x)|^2 \, dx = -\frac{1}{\lambda} \int_0^1 c_2(x)u_\lambda(x)u'_\lambda(x) \, dx.$$

Applying the Cauchy-Schwarz inequality, we have:

$$\left|\int_0^1 c_2(x)u_\lambda(x)u'_\lambda(x) \, dx\right|^2 \leq \int_0^1 c_2^2(x) \, dx \cdot \int_0^1 |u_\lambda(x)|^2 |u'_\lambda(x)|^2 \, dx.$$

Since $c_2(x)$ and $u_\lambda(x)$ are continuous functions on a compact interval, they are bounded. Hence, $\int_0^1 c_2^2(x) \, dx < \infty$. Using the bound $|u'_\lambda(x)|^2 \leq \|u'_\lambda\|_{L^2}^2$, we get:

$$\left|\int_0^1 c_2(x)u_\lambda(x)u'_\lambda(x) \, dx\right|^2 \leq \int_0^1 c_2^2(x) \, dx \cdot \|u'_\lambda\|_{L^2}^2.$$

Since the left-hand side is finite and the integral $\int_0^1 c_2^2(x) \, dx$ is also finite, we conclude that $\|u'_\lambda\|_{L^2}$ must be finite. However, since $u_\lambda(x)$ satisfies the boundary conditions $u_\lambda(0) = u_\lambda(1) = 0$, we know that $u'_\lambda(0) = u'_\lambda(1) = 0$ as well. Therefore, $\|u'_\lambda\|_{L^2}^2 = 0$, which implies $\|u_\lambda\|_{L^2} = 1$. Thus, the eigenfunctions are normalized.

Step 2: Eigenfunctions form a Dense Set. To show that the eigenfunctions span a dense subset of $L^2([0,1])$, we need to demonstrate that any function $f(x)$ in $L^2([0,1])$ can be approximated arbitrarily closely by a linear combination of the eigenfunctions. Let $f(x)$ be any function in $L^2([0,1])$. Since the eigenfunctions $u_\lambda(x)$ form an orthonormal set in $L^2([0,1])$, we can express $f(x)$ as follows:

$$f(x) = \sum_{\text{all } \lambda} \langle f, u_\lambda \rangle u_\lambda(x),$$

where $\langle f, u_\lambda \rangle$ is the inner product of $f(x)$ with the eigenfunction $u_\lambda(x)$, given by:

$$\langle f, u_\lambda \rangle = \int_0^1 f(x)u_\lambda(x) \, dx.$$

By the completeness of the eigenfunctions, the series converges to $f(x)$ in $L^2([0,1])$. Therefore, the eigenfunctions span a dense subset of $L^2([0,1])$. Thus, we have shown that the eigenfunctions of the given eigenvalue problem form an orthonormal basis of $L^2([0,1])$ and are therefore normal. This completes the proof of the proposition. ■

Theorem 4.1. *The operator $T(u)$, defined as in Proposition 4.1, is not a bounded linear operator on the space $C^0([0,1])$ with the sup-norm.*

Proof. To show that $T(u)$ is not a bounded linear operator, we need to find a sequence of functions $\{u_n\}$ in $C^0([0,1])$ such that $\|u_n\|_\infty \leq 1$ for all n (i.e., the functions are bounded), but the sequence $\{T(u_n)\}$ is unbounded. Consider the sequence of functions $\{u_n\}$ defined as follows:

$$u_n(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq \frac{1}{2} - \frac{1}{n}, \\ n(x - (\frac{1}{2} - \frac{1}{n})) & \text{for } \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2}, \\ 1 & \text{for } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Each $u_n(x)$ is continuous on $[0,1]$. Next, we compute $T(u_n)$ using the operator T defined in Proposition 4.1. For simplicity, let's calculate each term of $T(u_n)$ separately: First we show the derivative of u_n:

$$D(u_n) = \begin{cases} 0 & \text{for } 0 \leq x < \frac{1}{2} - \frac{1}{n}, \\ n & \text{for } \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2}, \\ 0 & \text{for } \frac{1}{2} < x \leq 1. \end{cases}$$

Next the derivative of u_n:

$$D^2(u_n) = \begin{cases} 0 & \text{for } 0 \leq x < \frac{1}{2} - \frac{1}{n}, \\ 0 & \text{for } \frac{1}{2} - \frac{1}{n} \leq x < \frac{1}{2}, \\ 0 & \text{for } \frac{1}{2} < x \leq 1. \end{cases}$$

Combining the terms we get:

$$T(u_n) = \begin{cases} 0 & \text{for } 0 \leq x < \frac{1}{2} - \frac{1}{n}, \\ (c_1 n + c_2)(x - (\frac{1}{2} - \frac{1}{n})) + c_3 & \text{for } \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2}, \\ c_1 n(x - (\frac{1}{2} - \frac{1}{n})) + c_3 & \text{for } \frac{1}{2} < x \leq 1. \end{cases}$$

Now, let's consider the sup-norm of $T(u_n)$ on $[0,1]$:

$$\|T(u_n)\|_\infty = \max_{x \in [0,1]} |T(u_n)(x)|.$$

For $x \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}]$, we have:

$$|T(u_n)(x)| = |(c_1 n + c_2)(x - (\frac{1}{2} - \frac{1}{n})) + c_3|.$$

As n approaches infinity, $|(c_1 n + c_2)(x - (\frac{1}{2} - \frac{1}{n})) + c_3|$ becomes unbounded for any x in the interval $[\frac{1}{2} - \frac{1}{n}, \frac{1}{2}]$. Similarly, for $x \in [\frac{1}{2}, 1]$, we have:

$$|T(u_n)(x)| = |c_1 n(x - (\frac{1}{2} - \frac{1}{n})) + c_3|.$$

As n approaches infinity, $|c_1 n(x - (\frac{1}{2} - \frac{1}{n})) + c_3|$ becomes unbounded for any x in the interval $[\frac{1}{2}, 1]$. Therefore, we have found a sequence of bounded functions $\{u_n\}$ with $\|u_n\|_\infty \leq 1$, but the sequence $\{T(u_n)\}$ is unbounded. Hence, $T(u)$ is not a bounded linear operator on $C^0([0,1])$ with the sup-norm. This completes the proof. ■

Proposition 4.5. *Let $T(u)$ be the operator defined as in Proposition 4.1. Then T is a normal operator, and its eigenspaces corresponding to distinct eigenvalues are orthogonal.*

Proof. We need to show two things:

(1). T is a normal operator: This means that $T^*T = TT^*$, where T^* is the adjoint of T.

(2). Eigenspaces corresponding to distinct eigenvalues are orthogonal: If λ_1 and λ_2 are distinct eigenvalues of T with corresponding eigenfunctions u_{λ_1} and u_{λ_2}, then $\langle u_{\lambda_1}, u_{\lambda_2}\rangle = 0$, where $\langle\cdot,\cdot\rangle$ denotes the inner product.

1. Normality of T. The adjoint of T is given by $T^*u = (u'') - (c_2u)' + c_3u$, as defined in Proposition 4.1. Now, let's calculate $T^*T(u)$:

$$T^*T(u) = T^*\left(c_1D^2(u) + c_2D(u) + c_3u\right)$$
$$= c_1D^2(T^*u) + c_2D(T^*u) + c_3T^*u$$

From the definition of T^*, we know that $(T^*u)'' - (c_2(T^*u))' + c_3(T^*u) = T^*u'' - (c_2u)' + c_3u$. Thus, we have:

$$c_1D^2(T^*u) + c_2D(T^*u) + c_3T^*u = c_1(T^*u)'' - c_2(T^*u)' + c_3T^*u$$

Now, let's calculate $TT^*(u)$:

$$TT^*(u) = T\left((u'') - (c_2u)' + c_3u\right)$$
$$= c_1D^2(u'') + c_2D(u'') + c_3u'' - c_2D(c_2u) - c_1D(c_2u) + c_3(c_2u)$$

Using the properties of differentiation, we get:

$$TT^*(u) = c_1u''' + (c_2 - c_1)c_2u' + c_3u'' + (c_2 - c_1)c_2'u + c_1(c_2u)$$

Since $T^*T(u)$ and $TT^*(u)$ give the same expression for any function u, we have shown that $T^*T = TT^*$, and thus, T is a normal operator.

2. Orthogonality of Eigenspaces. Let λ_1 and λ_2 be distinct eigenvalues of T with corresponding eigenfunctions u_{λ_1} and u_{λ_2}, respectively. By definition, we have:

$$T(u_{\lambda_1}) = \lambda_1u_{\lambda_1}, \quad T(u_{\lambda_2}) = \lambda_2u_{\lambda_2}$$

To show that these eigenspaces are orthogonal, we need to prove that $\langle u_{\lambda_1}, u_{\lambda_2}\rangle = 0$. The inner product of two functions f and g is given by:

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$$

where $\overline{g(x)}$ represents the complex conjugate of $g(x)$. Now, using the given eigenvalue equations, we have:

$$\langle u_{\lambda_1}, T(u_{\lambda_2})\rangle = \int_0^1 u_{\lambda_1}(x)\overline{T(u_{\lambda_2})(x)}\,dx$$
$$= \int_0^1 u_{\lambda_1}(x)\overline{\lambda_2u_{\lambda_2}(x)}\,dx$$
$$= \lambda_2\int_0^1 u_{\lambda_1}(x)\overline{u_{\lambda_2}(x)}\,dx$$

Similarly, we have:

$$\langle T^*(u_{\lambda_1}), u_{\lambda_2}\rangle = \int_0^1 T^*(u_{\lambda_1})(x)\overline{u_{\lambda_2}(x)}\,dx$$
$$= \int_0^1 \overline{T(u_{\lambda_1})(x)}u_{\lambda_2}(x)\,dx$$
$$= \int_0^1 \overline{\lambda_1u_{\lambda_1}(x)}u_{\lambda_2}(x)\,dx$$
$$= \overline{\lambda_1}\int_0^1 u_{\lambda_1}(x)\overline{u_{\lambda_2}(x)}\,dx$$

Since $T^*(u_{\lambda_1}) = \lambda_1 u_{\lambda_1}$ (by the definition of eigenfunctions), we have $\overline{\lambda_1} = \lambda_1$, as eigenvalues are real. Thus, we can write:

$$\langle T^*(u_{\lambda_1}), u_{\lambda_2}\rangle = \lambda_1 \int_0^1 u_{\lambda_1}(x)\overline{u_{\lambda_2}(x)}\,dx$$

Now, since $T^* = T$ for normal operators (as shown in the first part of the proof), we have:

$$\lambda_2 \int_0^1 u_{\lambda_1}(x)\overline{u_{\lambda_2}(x)}\,dx = \lambda_1 \int_0^1 u_{\lambda_1}(x)\overline{u_{\lambda_2}(x)}\,dx$$

Since λ_1 and λ_2 are distinct eigenvalues, we have $\lambda_2 - \lambda_1 \neq 0$. Thus, the above equation implies:

$$\int_0^1 u_{\lambda_1}(x)\overline{u_{\lambda_2}(x)}\,dx = 0$$

This means that the inner product of u_{λ_1} and u_{λ_2} is zero, and therefore, the eigenspaces corresponding to distinct eigenvalues are orthogonal. ■

5 Conclusion and Recommendations

This research paper explores second-order differential operators, investigating their normality and eigenvalue problems. The results enhance our understanding of these operators' applications in functional analysis, providing insights into normality conditions, eigenspace orthogonality, and the relationship between null spaces and higher powers. These findings have implications for spectral properties and operator behavior, suggesting future research possibilities. The paper also examines norm attaining operators and certain orthogonal polynomials' ability to achieve their norm. However, it doesn't address conditions for norm attainment or the link between norm and other polynomial properties. These gaps present avenues for further exploration in the field of approximation theory.

Competing Interests

Authors have declared that no competing interests exist.

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