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Solution to Fractional Schrödinger and Airy Differential Equations via Integral Transforms

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Abstract

Due to the need and the necessity to express a physical phenomenon in terms of an effective and comprehensive analytical form, this paper is devoted to study of Airy functions, which arise from the Airy differential equations, by means of integral transforms. Illustrative examples are also provided. The result reveals that the integral transforms are very useful tools to solve differential equations.

Keywords: Laplace transform, Fourier transform, L_2 – transform, airy functions, fractional airy differential equations, Schrödinger equation.

Mathematics Subject Classification2010: 33C47; 43A30; 34K37; 35R11.

1 Introduction

Airy differential equation named after British mathematician and astronomer George Biddell Airy (1801–1892) is a special DE in physics which is used in evaluation of diffraction of light near the caustic surface (such as rainbow). The Airy differential equation in fact is a special case of Schrödinger's equation for a particle confined within a triangular potential well and for a particle in a one-dimensional constant force field. The Airy function is also important in microscopy and astronomy: it describes the pattern due to diffraction and interference, produced by a point source of light (one which is smaller than the resolution limit of a microscope or telescope).

The Fourier and Laplace type integral transforms are wonderful alternative methods for solving different types of PDEs of fractional order. There are a lot of applications of PFDEs in the field of Visco elasticity as well.

In this work, the authors implemented Laplace integral transform method for solving fractional Airy equation which arise in applications. Several methods have been previously introduced to solve fractional differential equations (see [1,2,3,4,5,6,7,8,9]) however most of these methods are suitable for special types of fractional differential equations, mainly the linear ones with constant

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coefficients. In recent years, the implementations of extended G'/G – method for the solutions of non – linear evolution equations, non linear Klein - Gordon equations, Boussinesq equations have been well – established by notable researchers (see [10,11,12,13,14])

2 Definitions and Notations

Definition.2.1. Laplace transform of the function $f(t)$ is defined as follows

$$L\{f(t);t\to s\}=\int_0^\infty e^{-st}f(t)dt := F(s). \qquad (2.1)$$

If $L\{f(t)\}=F(s)$, then $L^{-1}\{F(s)\}$ is given by

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds, \qquad (2.2)$$

where $F(s)$ is analytic in the region $\mathrm{Re}(s)>c$.

Definition.2.2. For an arbitrary real number $\alpha>0$ $(n-1\le\alpha<n,\ n\in N)$ Caputo fractional derivative is given as

$${}_a^C D_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}\,dx.$$

Theorem.2.3. For $n-1<\alpha\le n$ one gets

$$L\{{}_0^C D_t^\alpha f(t)\}=s^\alpha F(s)-\sum_{k=0}^{n-1} s^{\alpha-k-1}f^{(k)}(0).$$

Proof. See [15].

Lemma.2.4. (Titchmarsh) Let $F(p)$ be an analytic function having no singularities in the cut plane $C\setminus R_-$. Assume that $\overline{F(p)}=F(\overline{p})$ and the limiting values

$$F^\pm(t)=\lim_{\phi\to\pi^-}F(te^{\pm i\phi}),\quad F^+(t)=\overline{F^-(t)},$$

exist for almost all

(i) $F(p) = o(1)$ for $|p| \to \infty$ and $F(p) = o(|p|^{-1})$ for $|p| \to 0$, uniformly in any sector $|\arg p| < \pi - \eta, \pi > \eta > 0$;

(ii) There exists $\varepsilon > 0$ such that for every $\pi - \varepsilon < \phi \le \pi$,

$$\frac{F(re^{\pm i\phi})}{1+r} \in L^1(R_+), \quad |F(re^{\pm i\phi})| \le a(r),$$

where $a(r)$ does not depend on ϕ and $a(r)e^{-\delta r} \in L^1(R_+)$ for any $\delta > 0$. Then, in the notation of the problem,

$$f(t) = L^{-1}[F(s)] = \frac{1}{\pi}\int_0^\infty \operatorname{Im}[F^-(\eta)]e^{-t\eta}d\eta.$$

Proof. See [16].

Example.2.5. Solve the following ODE under the given boundary condition

$$D_t^{2\alpha} y - kD_t^{\alpha} y = J_0(2\sqrt{t}), \quad y(0) = 0, y'(0) = 0, \ 0.5 < \alpha \le 1.$$

Solution. To solve the above ODE first we take Laplace transform of both sides of the equation

$$s^{2\alpha} Y(s) - ks^{\alpha} Y(s) = \frac{1}{s} e^{-\frac{1}{s}},$$

in which

$$Y(s) = L\{y(t); s\},$$

therefore

$$Y(s) = \frac{e^{-\frac{1}{s}}}{s^{\alpha+1}(s^{\alpha} - k)},$$

by using Titchmarsh theorem we have

$$L^{-1}\{\frac{1}{s^{\alpha} - k}; t\} = \frac{1}{\pi}\int_0^\infty e^{-rt} \operatorname{Im}\left\{\frac{1}{(re^{-i\pi})^{\alpha} - k}\right\} dr = \frac{1}{\pi}\int_0^\infty e^{-rt} \operatorname{Im}\left\{\frac{1}{(r^{\alpha}\cos\frac{\alpha\pi}{2} - k) + ir^{\alpha}\sin\frac{\alpha\pi}{2}}\right\} dr,$$

which can be rewritten in the form

$$L^{-1}\{\frac{1}{s^{\alpha}-k};t\}=-\frac{1}{\pi}\sin\frac{\alpha\pi}{2}\int_{0}^{\infty}e^{-rt}\frac{r^{\alpha}}{(r^{\alpha}-k\cos\frac{\alpha\pi}{2})^{2}+k^{2}(1+\cos^{2}\frac{\alpha\pi}{2})}dr$$

now considering Laplace transform of convolution of functions we get the following result

$$y(t)=-\frac{1}{\pi\Gamma(\alpha-1)}\sin\frac{\alpha\pi}{2}J_{\nu}(2\sqrt{t})*\int_{0}^{\infty}\frac{r^{\alpha}e^{-rt}}{(r^{\alpha}-k\cos\frac{\alpha\pi}{2})^{2}+k^{2}(1+\cos^{2}\frac{\alpha\pi}{2})}dr.$$

Definition.2.6. The differential operator δ called the δ-derivative is defined as

$$\delta_{t}=\frac{1}{t}\frac{d}{dt},\ \ \delta_{t}^{2}=\delta_{t}\delta_{t}=\frac{1}{t^{2}}\frac{d^{2}}{dt^{2}}-\frac{1}{t^{3}}\frac{d}{dt},$$

this definition can be extended to any positive integer power.

Lemma.2.7.(Schouten-Vanderpol) Consider a function $f(t)$ which has the Laplace transform $F(s)$ which is analytic in the half plane $\mathrm{Re}(s)>c$. If $q(s)$ is also analytic for $\mathrm{Re}(s)>c$, then the inverse of $F(q(s))$ is as follows

$$L^{-1}\{F(q(s));s\to t\}=\int_{0}^{\infty}f(\tau)\left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{-q(s)\tau}e^{ts}ds\right]d\tau.$$

Special Case: $q(s)=s^{\frac{3}{2}}$;

$$L^{-1}\{F(s^{\frac{3}{2}});s\to t\}=\int_{0}^{\infty}f(\tau)\left(\int_{0}^{\infty}e^{-\tau\eta^{\frac{3}{2}}}\sin(\tau\eta^{\frac{3}{2}})e^{-t\eta}d\eta\right)d\tau.$$

Proof: See [17].

Problem.2.8. Find inverse Laplace transform of the following function

$$F(s)=\frac{e^{-\lambda s^{\alpha}}}{s^{\beta}};\quad 0<\alpha<1,\quad 0\le\beta<1.$$

Solution. By using lemma 2.4 we have

$$L^{-1}\{F(s);t\} = f(t) = \frac{1}{\pi}\int_0^{\infty} \mathrm{Im}[\lim_{\phi\to\pi^-} F(\eta e^{-i\phi})] e^{-t\eta} d\eta,$$

substituting the function $F(s)$ in the above formula we get

$$L^{-1}\{\frac{e^{-\lambda s^{\alpha}}}{s^{\beta}};t\} = \frac{1}{\pi}\int_0^{\infty} \mathrm{Im}\left(\frac{e^{-\lambda(\eta e^{-i\pi})^{\alpha}}}{(\eta e^{-i\pi})^{\beta}}\right) e^{-t\eta} d\eta,$$

therefore, the result is obtained as

$$L^{-1}\{\frac{e^{-\lambda s^{\alpha}}}{s^{\beta}};t\} = \frac{1}{\pi}\int_0^{\infty} \frac{e^{-\lambda\eta^{\alpha}\cos\alpha\pi}}{\eta^{\beta}} \sin(\pi\beta + \lambda\eta^{\alpha}\sin\pi\alpha) e^{-t\eta} d\eta.$$

Special Case : $\beta = 0 \quad ,\alpha = \frac{1}{3}$;

Using Schouten-Vanderpol we have

$$L^{-1}\{F(s^{\frac{1}{3}});s \to t\} = \int_0^{\infty} f(\tau)\left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-s^{\frac{1}{3}}\tau} e^{ts} ds\right] d\tau.$$

Now it suffices to use Problem 2.7 for $\beta = 0, \alpha = \frac{1}{3}$ to get

$$L^{-1}\{e^{-s^{\frac{1}{3}}x};t\} = \frac{1}{\pi}\int_0^{\infty} e^{-\frac{1}{2}\tau\eta^{\frac{1}{3}}} \sin(\frac{\sqrt{3}}{2}\eta^{\frac{1}{3}}\tau) e^{-t\eta} d\eta,$$

it means that

$$L^{-1}\{F(s^{\frac{1}{3}});s \to t\} = \frac{1}{\pi}\int_0^{\infty} f(\tau)\left(\int_0^{\infty} e^{-t\eta-\frac{1}{2}\tau\eta^{\frac{1}{3}}} \sin(\frac{\sqrt{3}}{2}\eta^{\frac{1}{3}}\tau)\, d\eta\right) d\tau,$$

Lemma.2.9. We have the following integral representation for modified Bessel's function of the second kind

$$K_{\nu}(t)=\frac{1}{2}\left(\frac{z}{2}\right)^{\nu}\int_{0}^{+\infty}\exp\left\{-t-\frac{z^{2}}{4t}\right\}\frac{dt}{t^{\nu+1}}.$$

Proof. See [18].

Lemma.2.10. The following relationship holds true

$$\int_{0}^{+\infty} z^{\mu}K_{\nu}(z)dz = 2^{\mu-1}\Gamma(\frac{\mu+\nu+1}{2})\Gamma(\frac{\mu-\nu+1}{2}).$$

Proof. By substituting the above integral representation we have

$$\int_{0}^{+\infty} z^{\mu}K_{\nu}(z)dz = \int_{0}^{+\infty} z^{\mu}\left\{\frac{1}{2}\left(\frac{z}{2}\right)^{\nu}\int_{0}^{+\infty}\exp(-t-\frac{z^{2}}{4t})\frac{dt}{t^{\nu+1}}\right\}dz\,,$$

changing the order of integrals we get

$$\int_{0}^{+\infty} z^{\mu}K_{\nu}(z)dz = \left(\frac{1}{2}\right)^{\nu+1}\int_{0}^{+\infty} e^{-t}\left(\int_{0}^{+\infty} z^{\nu+\mu}e^{-\frac{z^{2}}{4t}}dz\right)\frac{dt}{t^{\nu+1}},$$

making a change of new variable $\frac{z^{2}}{4t}=w$, the above integral could be rewritten as below

$$\int_{0}^{+\infty} z^{\mu}K_{\nu}(z)dz = 2^{\mu-1}\int_{0}^{+\infty} w^{\frac{\mu+\nu-1}{2}}e^{-w}dw\int_{0}^{+\infty} t^{\frac{\mu-\nu-1}{2}}e^{-t}dt\,,$$

which can be evaluated by using definition of Laplace transform as follows

$$\int_{0}^{+\infty} z^{\mu}K_{\nu}(z)dz = 2^{\mu-1}\Gamma(\frac{\mu+\nu+1}{2})\Gamma(\frac{\mu-\nu+1}{2}).$$

Example.2.11. Prove that

$$\int_{0}^{+\infty} K_{\frac{1}{2}}(az)dz = \frac{\pi}{a\sqrt{2}},\quad \int_{0}^{+\infty} K_{\frac{1}{3}}(az)dz = \frac{\pi}{a}.$$

Solution. By making a change of variable $az = u$ and then using the above lemma for $\mu = 0, \nu = \frac{1}{2}$, we have

$$\int_0^{+\infty} K_{\frac{1}{2}}(az)dz = \frac{1}{a}\int_0^{+\infty} K_{\frac{1}{2}}(u)du = \frac{1}{2a}\Gamma(\frac{3}{4})\Gamma(\frac{1}{4}) = \frac{1}{2a}\Gamma(1-\frac{1}{4})\Gamma(\frac{1}{4}) = \frac{1}{2a}\cdot\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{a\sqrt{2}},$$

and again let $\mu = 0, \nu = \frac{1}{3}$ to get

$$\int_0^{+\infty} K_{\frac{1}{3}}(az)dz = \frac{1}{a}\int_0^{+\infty} K_{\frac{1}{3}}(u)du = \frac{1}{2a}\Gamma(\frac{2}{3})\Gamma(\frac{1}{3}) = \frac{1}{2a}\Gamma(1-\frac{1}{3})\Gamma(\frac{1}{3}) = \frac{1}{2a}\cdot\frac{\pi}{\sin\frac{\pi}{6}} = \frac{\pi}{a}.$$

Generally it can be shown that

$$\int_0^{+\infty} K_{\frac{1}{n}}(az)dz = \frac{1}{2a}\cdot\frac{\pi}{\sin\frac{\pi}{n}}\cdot$$

Theorem.2.12. (Buschman) If the functions $f(t), g(t), k(t)$ be analytical and real on $(0,\infty)$ such that $g(0) = 0$ and $g(\infty) = \infty$. Then

$$L\{k(t)f[g(t)]\} = \int_0^{+\infty} R(s,u)F(u)du.$$

in which

$$Q(s,p) = \int_0^{+\infty} e^{-pu}R(s,u)du = e^{-sh(p)}k[h(p)]h'(p), \quad h = g^{-1},$$

Proof. See [19].

Example.2.13. The following relationship holds true

$$L\{f(\frac{1}{t})\} = \frac{1}{\sqrt{s}}\int_0^{+\infty} \sqrt{u}J_1(2\sqrt{su})F(u)du.$$

Solution. It suffices to take $k(t)=1, g(t)=\frac{1}{t}$ in Buschman theorem, then

$$R(s,u)=L^{-1}\{-\frac{e^{-\frac{s}{p}}}{p^2}\}=-\sqrt{\frac{u}{s}}J_1(2\sqrt{su}),$$

and therefore

$$L\{f(\frac{1}{t})\}=\frac{1}{\sqrt{s}}\int_0^{+\infty}\sqrt{u}J_1(2\sqrt{su})F(u)du.$$

Example.2.14. Let $f(t)=\ln t$ then $F(s)=L\{f(t)\}=-\frac{\gamma+\ln s}{s}$. And also $L\{\ln\frac{1}{t}\}=-L\{\ln t\}=\frac{\gamma+\ln s}{s}$. Therefore using the above lemma one gets that

$$\frac{\gamma+\ln s}{s}=-\frac{1}{\sqrt{s}}\int_0^{\infty}J_1(2\sqrt{su})\frac{\gamma+\ln u}{\sqrt{u}}du\cdot$$

Especially for $s=1$ we have

$$\gamma=-\int_0^{\infty}J_1(2\sqrt{u})\frac{\gamma+\ln u}{\sqrt{u}}du\cdot$$

Example.2.15. Consider the function $f(t)=\frac{1}{2}te^{-\lambda t}$ then $F(s)=L\{f(t)\}=\frac{1}{2(s+\lambda)^2}$.

On the other hand $f(\frac{1}{t})=\frac{1}{2t}e^{-\frac{\lambda}{t}}$ and $L\{f(\frac{1}{t})\}=K_0(2\sqrt{\lambda s})$. Now using lemma 2.13 we have

$$K_0(2\sqrt{\lambda s})=\frac{1}{2\sqrt{s}}\int_0^{\infty}\sqrt{u}J_1(2\sqrt{su})\frac{1}{(u+\lambda)^2}du,$$

and especially for $s=1$ we get the following integral representation for $K_0(2\sqrt{\lambda})$

$$K_0(2\sqrt{\lambda}) = \frac{1}{2}\int_0^{\infty} \frac{\sqrt{u}J_1(2\sqrt{u})}{(u+\lambda)^2}du \cdot$$

3 Integral Transforms for Fractional Differential Equations

The classical integer order constitutive differential equations are approximations to every - thing as “point” quantity in time or in space. The classical integer order methods do not thus take into account the space history or time history and therefore cannot represent the natural laws close to reality. Fractional calculus does take of all these reality and therefore is more appropriate for representation of natural phenomena. Differential equations of fractional order appear more and more frequently in various research areas of science and engineering. An effective method for solving such equations is needed. The method of Fourier, Laplace, L_2 - transforms technique gives almost a unified approach to solve the fractional differential equations.

3.1 Airy Differential Equations

George Biddell Airy (1801–1892) was particularly involved in optics for this reason, he was also interested in the calculation of light intensity in the neighborhood of a caustic (see [20,21]). For this purpose, he introduced the function defined by the integral

$$W(m) = \int_0^{+\infty} \cos\left[\frac{\pi}{2}(w^3 - mw)\right]dw,$$

which is the solution of the following differential equation

$$W'' + \frac{\pi^2}{12}mW = 0.$$

In 1928 Jeffreys introduced the notation used nowadays

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it(x+\frac{t^2}{3})}dt = \frac{1}{\pi}\int_0^{+\infty}\cos\left(\frac{t^3}{3}+xt\right)dt, \qquad (3.1)$$

which is the solution of the following homogeneous ODE called Airy ODE

$$y'' - xy = 0. \qquad (3.2)$$

The following Bairy function is another solution for Airy DE which differs from Airy function in phase by $\dfrac{\pi}{2}$

$$Bi(x) = \frac{1}{\pi}\int_0^{+\infty}\left[\exp(-\frac{t^3}{3}+xt)+\sin(\frac{t^3}{3}+xt)\right]dt.$$

In some texts the Airy differential equation is considered as

$$y'' + f(x)y = 0,$$

in which $f(x)$ is any function having expansion in a neighborhood of a point $x = x_0$ with $f'(x_0) \neq 0$. The values of Ai(x) and Bi(x) and their derivatives at x = 0 are given by (see [21])

$$Ai(0) = \frac{1}{3^{\frac{2}{3}}\Gamma(\frac{2}{3})}; \quad Bi(0) = \frac{1}{3^{\frac{1}{6}}\Gamma(\frac{2}{3})}; \quad Ai'(0) = -\frac{1}{3^{\frac{1}{3}}\Gamma(\frac{1}{3})}; \quad Bi'(0) = \frac{3^{\frac{1}{6}}}{\Gamma(\frac{1}{3})}\cdot$$

Lemma.3.1.1. The following integral relationship holds true

$$Ai(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}}K_{\frac{1}{3}}(\frac{2}{3}x^{\frac{3}{2}}).$$

Proof. See [22].

Lemma.3.1.2. The following relationship holds true

$$\int_{-\infty}^{\infty} Ai(x)\,dx = 1\cdot$$

Proof. Taking Fourier transform of the Airy function we have

$$F\{Ai(x);s\} = \int_{-\infty}^{\infty} e^{-isx} Ai(x)dx = \int_{-\infty}^{\infty} e^{-isx}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixt+i\frac{t^3}{3}}dt\right)dx,$$

changing the order of integrals and considering Fourier transform for fundamental functions, we get

$$F\{Ai(x);s\} = \int_{-\infty}^{\infty} e^{i\frac{t^3}{3}}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix(t-s)}dx\right)dt = \int_{-\infty}^{\infty} e^{i\frac{t^3}{3}}\delta(s-t)dt = e^{i\frac{s^3}{3}},$$

now it suffices to let $s=0$ to get the desired result

$$\int_{-\infty}^{\infty} Ai(x)\,dx = 1\cdot$$

Lemma.3.1.3. The following relationship holds true

$$\int_{0}^{\infty} Ai(x)dx = \frac{1}{3}.$$

Proof. By using lemma 3.1.1 we can write

$$\int_{0}^{\infty} Ai(x)dx = \frac{1}{\pi\sqrt{3}}\int_{0}^{\infty}\sqrt{x}\,K_{\frac{1}{3}}(\frac{2}{3}x^{\frac{3}{2}})dx$$

making a change of variable $z=\frac{2}{3}x^{\frac{3}{2}}$ we get

$$\int_{0}^{\infty} Ai(x)dx = \frac{1}{\pi\sqrt{3}}\int_{0}^{\infty} K_{\frac{1}{3}}(z)dz\,,$$

now using lemma 2.10 for $\mu=0,\nu=\frac{1}{3}$, the following result will be obtained

$$\int_{0}^{\infty} Ai(x)dx = \frac{1}{\pi\sqrt{3}}\frac{1}{2}\Gamma(\frac{2}{3})\Gamma(\frac{1}{3}) = \frac{1}{\pi\sqrt{3}}\frac{1}{2}\frac{\pi}{\sin\frac{\pi}{3}} = \frac{1}{3}\cdot$$

Lemma.3.1.4. The following relationship holds true

$$\int_{0}^{\infty} Ai^{2}(t)dt = \frac{1}{3^{\frac{2}{3}}\Gamma^{2}(\frac{1}{3})}.$$

Note that, $Ai^2(t)$ is not square integrable over whole real line R.

Proof. Consider the function

$$I(x) = \int_x^\infty Ai^2(t)dt.$$

Integrating by parts we have

$$I(x) = tAi^2(t)\Big|_x^\infty - \int_x^\infty 2tAi(t)Ai'(t)dt,$$

but we know that the Airy function is a solution of Airy DE therefore one can rewrite the above relationship as below

$$I(x) = tAi^2(t)\Big|_x^\infty - 2\int_x^\infty Ai''(t)Ai'(t)dt = tAi^2(t)\Big|_x^\infty - Ai'^2(t)\Big|_x^\infty,$$

on the other hand from the following integral representation (see [22])

$$Ai(x) = \frac{\sqrt{3}}{2\pi}\int_0^\infty e^{-\frac{t^3}{3}-\frac{x^3}{3t^3}}dt,$$

we get

$$\lim_{x\to\infty} xAi^2(x) = 0, \quad \lim_{x\to\infty} Ai'(x) = 0,$$

it means that

$$I(x) = -xAi^2(x) + Ai'^2(x),$$

therefore we have

$$I(0) = \int_0^\infty Ai^2(t)dt = Ai'^2(0) = \frac{1}{3^{\frac{2}{3}}\Gamma^2(\frac{1}{3})}\cdot$$

Lemma.3.1.5. The following integral representation holds true

$$1 - K_\nu(\xi) = \int_0^a \frac{t^{\nu+1}}{(a^2-t^2)^{\frac{\nu}{2}+1}} J_\nu\left(\frac{\xi t}{\sqrt{a^2-t^2}}\right)dt,$$

$$2-Ai(x)=\frac{1}{\pi}\sqrt{\frac{x}{3}}\int_0^\infty u^{\frac{5}{3}}J_{\frac{2}{3}}(\frac{2}{3}ux^{\frac{3}{2}})\frac{du}{1+u^2}.$$

Proof . 1) Making a change of variable $w^2=u$ we can write

$$\int_0^\infty \frac{w^{\nu+1}J_\nu(\xi w)}{w^2+1}dw=\frac{1}{2}\int_0^\infty \frac{u^{\frac{\nu}{2}}J_\nu(\xi\sqrt{u})}{u+1}du,$$

which is in the form of Stieltjes transform, on the other hand the Stieltjes transform is the second iteration of Laplace transform therefore

$$\int_0^\infty \frac{w^{\nu+1}J_\nu(\xi w)}{w^2+1}dw=\frac{1}{2}L\{L\{u^{\frac{\nu}{2}}J_\nu(\xi\sqrt{u});u\to s\};s\to 1\}$$

$$=\frac{1}{2}L\{(\frac{\xi}{2})^\nu s^{-\nu-1}e^{-\frac{\xi^2}{4s}};s\to 1\}=\frac{\xi^\nu}{2^{\nu+1}}\times 2(\frac{\xi^2}{4})^{-\frac{\nu}{2}}K_{-\nu}(\xi)=K_\nu(\xi).$$

now it suffices to set $w=\frac{t}{\sqrt{a^2-t^2}}$ to get the following relationship

$$K_\nu(\xi)=\int_0^a \frac{t^{\nu+1}}{(a^2-t^2)^{\frac{\nu}{2}+1}}J_\nu(\frac{\xi t}{\sqrt{a^2-t^2}})dt.$$

2) From the previous part, we know that

$$K_\nu(\xi)=\int_0^a \frac{t^{\nu+1}}{(a^2-t^2)^{\frac{\nu}{2}+1}}J_\nu(\frac{\xi t}{\sqrt{a^2-t^2}})dt,$$

making a change of new variable $\xi=\frac{2}{3}x^{\frac{3}{2}}$ for $\nu=\frac{1}{3}$ we have

$$K_{\frac{1}{3}}(\frac{2}{3}x^{\frac{3}{2}})=\int_0^a \frac{t^{\frac{4}{3}}}{(a^2-t^2)^{\frac{7}{6}}}J_{\frac{1}{3}}(\frac{2x^{\frac{3}{2}}t}{3\sqrt{a^2-t^2}})dt,$$

therefore

$$Ai(x)=\frac{1}{\pi}\sqrt{\frac{x}{3}}K_{\frac{1}{3}}(\frac{2}{3}x^{\frac{3}{2}})=\frac{1}{\pi}\sqrt{\frac{x}{3}}\int_0^a\frac{t^{\frac{4}{3}}}{(a^2-t^2)^{\frac{7}{6}}}J_{\frac{1}{3}}(\frac{2x^{\frac{3}{2}}t}{3\sqrt{a^2-t^2}})dt.$$

making a new change of variable $\frac{t}{\sqrt{a^2-t^2}}=u$ we will have

$$Ai(x)=\frac{1}{\pi}\sqrt{\frac{x}{3}}\int_0^\infty u^{\frac{4}{3}}J_{\frac{1}{3}}(\frac{2}{3}ux^{\frac{3}{2}})\frac{du}{1+u^2}\cdot$$

Definition.3.1.6. Let y_1, y_2 be differentiable functions. The Wronskian $W\,(y_1, y_2)$ associated to y_1, y_2 is defined as

$$W\,(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_2y_1'.$$

Theorem.3.1.7. (Abel) If y_1, y_2 are two solutions to

$$y''+p(x)y'+q(x)y=0,$$

then the Wronskian of the two solutions is

$$W(y_1,y_2)(x)=W(y_1(x_0),y_2(x_0))\,e^{-\int_{x_0}^{x}p(t)dt},$$

for some x_0.

Proof. See [23].

Lemma.3.1.8. The following relationship holds true

$$Ai(x)Bi'(x)-Bi(x)Ai'(x)=\frac{1}{\pi}\cdot$$

Proof. By using theorem 3.1.7 for Airy differential equation and $p(x)=0$ then , for $x_0=0$ we have

$$W\,(Ai(x),Bi(x))=Ai(0)Bi'(0)-Bi(0)Ai'(0)=\frac{1}{\pi}\cdot$$

3.2 Solution to Airy DE by using L_2 - Integral Transform

Consider a free q charged particle, moving on the x axis plunged into a uniform electric field $\vec{\varepsilon}$. This particle is submitted to the force $\vec{F} = q\vec{\varepsilon}$ and its potential energy is $U = -Fx$. So the ***Schrödinger*** equation is checked by the wave function of the particle

$$\frac{d^2\psi}{dx^2} + \frac{2m}{h^2}(E + Fx)\psi = 0,$$

where E is the total energy of the particle. Let us perform the change of variable

$$\xi = \left(x + \frac{E}{F}\right)\left(\frac{2mF}{h^2}\right)^{\frac{1}{3}},$$

where ξ is a one dimensional variable. Then the Schrödinger equation is reduced to the Airy equation (see [22])

$$\frac{d^2\psi}{d\xi^2} + \xi\psi = 0 \cdot$$

Definition.3.2.1. The Laplace-type integral transform called L_2 - transform was introduced by Yurekli and Sadek in [24], as

$$L_2\{f(t);s\} = \int_0^\infty t \exp(-s^2t^2) f(t)dt. \qquad (3.2.1)$$

Although the L_2 - transform is not nearly as versatile in applications as are the Fourier and Laplace transform, there are some areas of application where it can be a useful tool. In particular, it is useful in the calculation of certain integrals, solving some special integral and differential equations. If we make a change of variables in the right-hand side of the above integral (3.2.1), we get

$$L_2\{f(t);s\} = \frac{1}{2}\int_0^\infty e^{-ts^2} f(\sqrt{t})dt.$$

We have the following relationship between the Laplace - transform and the L_2 -transform

$$L_2\{f(t);s\} = \frac{1}{2}L\{f(\sqrt{t});s^2\}.$$

Theorem.3.2.2. If $f, f', \ldots, f^{(n-1)}$ are all continuous functions with piecewise continuous derivative $f^{(n)}$ on the interval $t \geq 0$, and if all functions are of exponential order $\exp(c^2 t^2)$ as $t \to \infty$ for some constant c then for any $n = 1, 2, 3, \ldots$ the following relationships hold true

$$1. \qquad L_2\{\delta_t^n f(t); s\} = 2^n s^{2n} L_2\{f(t); s\} - 2^{n-1} s^{2(n-1)} f(0^+) - 2^{n-2} s^{2(n-2)} (\delta_t f)(0^+) - \cdots - (\delta_t^{n-1} f)(0^+),$$

$$2. \qquad L_2\{t^{2n} f(t); s\} = \frac{(-1)^n}{2^n} \delta_s^n L_2\{f(t); s\}.$$

Proof. See [25,26].

Example.3.2.3. We have the following relationships

$$1- \quad L_2\{\delta(t-a); s\} = a \exp(-a^2 s^2),$$

$$2- \quad L_2\{e^{-at}; s\} = \frac{1}{2s^2} - \frac{a\sqrt{\pi}}{4s^3} \exp\left(\frac{a^2}{4s^2}\right) Erfc\left(\frac{a}{2s}\right),$$

$$3- \quad L_2\{Erf(at); s\} = \frac{a}{2s^2\sqrt{s^2 + a^2}},$$

$$4- \quad L_2\{t^n; s\} = \frac{\Gamma\left(\frac{n}{2} + 1\right)}{2s^{n+2}}.$$

Proof. See [8].

Theorem 3.2.4. (Main theorem) Let us consider differential equation

$$y'' - ax^{p-1} y = 0; \quad a > 0. p \in N$$

It has the following formal solution

$$y(x) = A_1 \sqrt{x}\, I_{\frac{1}{p+1}}\left(\frac{2\sqrt{a}}{p+1} x^{\frac{p+1}{2}}\right) + A_2 \sqrt{x}\, I_{-\frac{1}{p+1}}\left(\frac{2\sqrt{a}}{p+1} x^{\frac{p+1}{2}}\right).$$

Proof. Making a change of variable $y = x^{\frac{1}{2}} v(x)$ we get the following DE

$$x^2 v''(x) + x v'(x) - (ax^{p+1} + \frac{1}{4}) v(x) = 0,$$

again introducing a new variable $z=\sqrt{a}x^{\frac{p+1}{2}}$ and manipulating we will have

$$z^2v''(z)+zv'(z)-\left(\frac{2}{p+1}\right)^2(z^2+\frac{1}{4})v(z)=0,$$

making change of variables $\frac{2}{p+1}z=t$ we have

$$t^2v''(t)+tv'(t)-(t^2+\frac{1}{(p+1)^2})v(t)=0,$$

which is the Bessel differential equation. Now let $v(t)=t^{-\frac{1}{p+1}}u(t)$ to get the following relationship, and after simplifying

$$t^{-\frac{1}{p+1}+2}u''(t)-\left(\frac{2}{p+1}-1\right)t^{-\frac{1}{p+1}+1}u'(t)-t^{-\frac{1}{p+1}+2}u(t)=0.$$

Now let us divide both sides of the above equation by $t^{-\frac{1}{p+1}+2}$ to get

$$\left[u''(t)-\frac{1}{t}u'(t)\right]-2(\frac{1}{p+1}-1)\frac{1}{t}u'(t)-u(t)=0,$$

by using the definition of δ-derivative for L_2- transform, one can rewrite the above equation as below

$$t^2\delta_t^2u(t)-2(\frac{1}{p+1}-1)\delta_tu(t)-u(t)=0,$$

taking L_2 transform of both sides of the above relationship and making use of theorem 3.2.2 we obtain

$$-\frac{1}{2}\delta_s[4s^4U(s)-2s^2u(0^+)-(\delta_tu)(0^+)]-2(\frac{1}{p+1}-1)[2s^2U(s)-u(0^+)]-U(s)=0,$$

in which $U(s)=L_2\{u(t);s\}$. The above relationship may be rewritten as follows

$$2s^3U'(s)+\left[4s^2(\frac{1}{p+1}+1)+1\right]U(s)=\frac{2}{p+1}u(0^+).$$

Assuming $u(0^+)=0$ we have

$$\frac{U'(s)}{U(s)}=-\frac{1}{2s^3}-\frac{2(\frac{1}{p+1}+1)}{s},$$

integrating the above relationship we can write

$$\ln U(s)=\frac{1}{4s^2}-2(\frac{1}{p+1}+1)\ln s+\ln c,$$

or

$$U(s)=Cs^{-2(\frac{1}{p+1}+1)}\exp(\frac{1}{4s^2})\cdot$$

Now we substitute the function $e^{\frac{1}{4s^2}}$ by its Taylor series expansion

$$U(s)=C\sum_{n=0}^{\infty}\frac{1}{n!2^{2n}s^{2n}}s^{-2(\frac{1}{p+1}+1)}=C\sum_{n=0}^{\infty}\frac{1}{n!2^{2n}}\cdot\frac{1}{s^{2(n+\frac{1}{p+1}+1)}},$$

taking inverse L_2 -transform to have

$$u(t)=C\sum_{n=0}^{\infty}\frac{1}{n!2^{2n}}\cdot\frac{2t^{2(n+\frac{1}{p+1})}}{\Gamma(n+1+\frac{1}{p+1})}.$$

Now it remains to move backward over all of the changing variables, first $v(t)=t^{-\frac{1}{p+1}}u(t)$

$$v(t)=2Ct^{\frac{1}{p+1}}\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+1+\frac{1}{p+1})}\cdot\left(\frac{t}{2}\right)^{2n},$$

second $t=\frac{2}{p+1}z$ then

$$v(z)=2C\left(\frac{2}{p}z\right)^{\frac{1}{p+1}}\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+1+\frac{1}{p+1})}\cdot\left(\frac{z}{p}\right)^{2n},$$

third $z=\sqrt{a}x^{\frac{p+1}{2}}$ therefore

$$v(x)=2C\left(\frac{2}{p}\sqrt{a}x^{\frac{p+1}{2}}\right)^{\frac{1}{p+1}}\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+1+\frac{1}{p+1})}\cdot\left(\sqrt{a}\,\frac{x^{\frac{p+1}{2}}}{p}\right)^{2n},$$

and finally $y=x^{\frac{1}{2}}v(x)$

$$y(x)=2C\sqrt{x}\left(\frac{2\sqrt{a}}{p+1}x^{\frac{p+1}{2}}\right)^{\frac{1}{p+1}}\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+1+\frac{1}{p+1})}\cdot\left(\sqrt{a}\,\frac{x^{\frac{p+1}{2}}}{p+1}\right)^{2n}.$$

Without loss of generality and because of having elegant relationship let us consider $C=2^{-\frac{1}{p+1}-1}$ to have

$$y(x)=\sqrt{x}\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+1+\frac{1}{p+1})}\cdot\left(\sqrt{a}\,\frac{x^{\frac{p+1}{2}}}{p+1}\right)^{2n+\frac{1}{p+1}},$$

which may be re written as

$$y(x)=A_1\sqrt{x}\,I_{\frac{1}{p+1}}(\frac{2\sqrt{a}}{p+1}x^{\frac{p+1}{2}})+A_2\sqrt{x}\,I_{-\frac{1}{p+1}}(\frac{2\sqrt{a}}{p+1}x^{\frac{p+1}{2}}).$$

Airy differential equation arises when consider $p=2, a=1$ then

$$y''-xy=0,$$

and the solution is

$$y(x)=A_1\sqrt{x}\,I_{\frac{1}{3}}(\frac{2}{3}x^{\frac{3}{2}})+A_2\sqrt{x}\,I_{-\frac{1}{3}}(\frac{2}{3}x^{\frac{3}{2}})=\frac{\sqrt{3}}{2}(A_1+A_2)Bi(x)+\frac{3}{2}(A_2-A_1)Ai(x).$$

3.3 Solution to Airy FDE via Fourier Transform

Definition.3.3.1. The Fourier transform of the function $f(x)$ is defined as following

$$F(\alpha)=F\{f(x);x\to\alpha\}=\int_{-\infty}^{+\infty}f(x)e^{-i\alpha x}dx,$$

if the integral exists. The inverse of Fourier transform is

$$F^{-1}\{F(\alpha);\alpha \to x\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\alpha)e^{i\alpha x}\,d\alpha.$$

Problem.3.3.2. Solve the following Airy differential equation

$$y'' - kxy = 0; \quad x \in R, \lim_{x\to\pm\infty} y(x) = \lim_{x\to\pm\infty} y'(x) = 0.$$

Solution. Taking Fourier transform of both sides of the above DE we have

$$\alpha^2 Y(\alpha) + ikY'(\alpha) = 0,$$

we solve the above first order ODE to get

$$Y(\alpha) = C\exp(i\frac{\alpha^3}{3k}),$$

taking inverse Fourier transform the following relationship is obtained

$$y(x) = \frac{C}{2\pi}\int_{-\infty}^{+\infty} e^{i\alpha x}\exp(i\frac{\alpha^3}{3k})d\alpha = \frac{C}{2\pi}\int_{-\infty}^{+\infty} e^{i\alpha(x+\frac{\alpha^2}{3k})}d\alpha = \frac{C}{\pi}\int_{0}^{+\infty}\cos\alpha(x+\frac{\alpha^2}{3k})d\alpha.$$

Lemma.3.3.3. The following relationship holds true

$$F\{Ai(t); t \to w\} = \exp(i\frac{w^3}{3}).$$

Proof. See [22].

Lemma.3.3.4. The following relationship holds true (see [27])

$$\int_{-\infty}^{+\infty} Ai(t+x)Ai(t+y)dt = \delta(x-y). \qquad (3.3.1)$$

Proof. Using Parseval identity, we substitute the Airy functions in the above formula by their Fourier transforms

$$\int_{-\infty}^{+\infty} Ai(t+x)Ai(t+y)dt = \int_{-\infty}^{+\infty} e^{iwx}Ai_F(w)e^{-iwy}\overline{Ai_F(w)}dw,$$

in which $Ai_F(w) = F\{Ai(t);t \to w\}$ and the bar sign above it indicates the complex conjugate. By using lemma 3.3.3 we have

$$\int_{-\infty}^{+\infty} Ai(t+x)Ai(t+y)dt = \int_{-\infty}^{+\infty} e^{iwx} e^{i\frac{w^3}{3}} e^{-iwy} e^{-i\frac{w^3}{3}} dw = \int_{-\infty}^{+\infty} e^{iw(x-y)} dw = \delta(x-y).$$

Case: Let $x = y = 0$ then we have

$$\int_{-\infty}^{\infty} Ai^2(t)dt = +\infty.$$

Problem.3.3.5. Solve the following Airy differential equation

$$y'' + (\lambda + \mu x)y = 0, \qquad y(x) \in S(R),$$

in which $S(R)$ is the Schwartz space.

Solution. Taking Fourier transform of the above DE and simplifying we will have

$$\frac{Y'(w)}{Y(w)} = \frac{-w^2 + \lambda}{i\mu},$$

in which $Y(w) = F\{y(x);w\}$. Now solve the above DE of the first order to get the following result

$$Y(w) = Ae^{-\frac{i}{\mu}(\lambda w + \frac{w^3}{3})},$$

it suffices to use the inverse Fourier transform (without loss of generality let $A = 1$)

$$y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\frac{\lambda w}{\mu} - wx + \frac{w^3}{3\mu})} dw = \frac{1}{\pi}\int_0^{\infty} \cos(\frac{\lambda w}{\mu} - wx + \frac{w^3}{3\mu})dw .$$

Problem.3.3.6. Consider the following fractional Airy differential equation which is a generalization of the previous DE

$$D^{2\alpha} y + (\lambda + \mu x)y = 0, \qquad y(x) \in S(R).$$

Solution. Again using Fourier transform we have

$$\frac{Y'(w)}{Y(w)} = -\frac{i}{\mu}(\lambda + (iw)^{2\alpha}),$$

which can be rewritten as below

$$\frac{Y'(w)}{Y(w)} = -\frac{i}{\mu}(\lambda + w^{2\alpha}e^{i\alpha\pi}),$$

it means that

$$Y(w) = Ae^{-\sin\alpha\pi}e^{-\frac{i}{\mu}(\lambda w - \frac{w^{2\alpha+1}}{2\alpha+1}\cos\alpha\pi)}.$$

Taking inverse Fourier transform we get the following result

$$y(x) = \frac{Ae^{-\sin\alpha\pi}}{2\pi}\int_{-\infty}^{\infty} e^{i(xw - \frac{\lambda}{\mu}w + \frac{1}{\mu}\cos\alpha\pi\frac{w^{2\alpha+1}}{2\alpha+1})}dw.$$

One could take $A = \alpha = 1$ to get the solution obtained in the previous problem.

Problem.3.3.7. Consider the following Airy differential equation of $(n+1)-$th order

$$y^{(n+1)} - \lambda xy = 0, \qquad y \in S(R), \quad \lambda \in R.$$

Solution. Taking Fourier transform we have

$$\frac{Y'(w)}{Y(w)} = \frac{i^n w^{n+1}}{\lambda},$$

solving the above differential equation to get

$$Y(w) = Ae^{\frac{i^n}{\lambda}(\frac{w^{n+2}}{n+2})},$$

taking inverse Fourier transform we will have

$$y(x) = \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{i(xw - \frac{i^{n+1}w^{n+2}}{\lambda(n+2)})}dw = \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{i(xw - \frac{w^{n+2}}{\lambda(n+2)}e^{i(n+1)\frac{\pi}{2}})}dw.$$

It can be rewritten as below

$$y(x) = \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{-\sin\frac{\pi(n+1)}{2}\frac{w^{n+2}}{n+2}} e^{i(xw-\cos\frac{\pi(n+1)}{2}\frac{w^{n+2}}{n+2})} dw\,.$$

Case: Let $n = 1, A = 1, \lambda = 1$ to get

$$Ai(x) = \frac{1}{\pi}\int_{0}^{\infty} \cos(xw + \frac{w^3}{3}) dw\,.$$

Problem.3.3.8. Solve the following differential equation

$$y''' + \lambda x y' + \mu y = 0, \quad \lim_{x\to\pm\infty} y(x) = \lim_{x\to\pm\infty} y'(x) = \lim_{x\to\pm\infty} y''(x) = 0\cdot$$

Solution. Taking Fourier integral transform we will have

$$(iw)^3 Y(w) - \lambda(wY'(w) + Y(w)) + \mu Y(w) = 0,$$

or

$$\frac{Y'(w)}{Y(w)} = -\frac{iw^3 + \lambda - \mu}{\lambda w},$$

the solution of the above differential equation is

$$Y(w) = Ae^{-i\frac{w^3}{3\lambda}}(w^{\frac{\mu}{\lambda}} - w)\cdot$$

Therefore taking inverse Fourier transform we have

$$y(x) = \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{iw(x-\frac{w^2}{3\lambda})}(w^{\frac{\mu}{\lambda}} - w) dw\,.$$

3.4 Solution to Airy FDE by means of Laplace Transform

Problem.3.4.1. Consider a very thin homogeneous rod, which could be treated as a one dimensional system, of length L placed in the ground and in equilibrium. Assume that the rod is free at its top. Then the horizontal variation of it in x direction is expressed by the following differential equation

$$IE\, x''' = -q(L - z)x',$$

in which I is the moment of inertia, E the Young modulus, q the weight per unit length and z the vertical position. Now let $x'(z) = u(z), \xi = \left(\frac{q}{EI}\right)^{\frac{1}{3}}(z - L)$ to have

$$u''(\xi) - \xi u(\xi) = 0,$$

which is an Airy differential equation. Note that the lower end is fixed, $x(0) = x'(0) = x''(0) = 0$ or $u(0) = u'(0) = 0$.

Taking Laplace transform with respect to the variable ξ of the last differential equation we have

$$s^2 U(s) - sx(0^+) - x'(0) + U'(s) = 0,$$

in which $U(s) = L\{u(\xi); \xi \to s\}$. Therefore we get the following ODE of the first kind

$$U'(s) + s^2 U(s) = 0. \qquad (3.4.3)$$

The solution of the homogeneous differential equation is obtained as below

$$U(s) = Ae^{-\frac{s^3}{3}},$$

therefore using inverse Laplace transform we have

$$u(\xi) = \frac{A}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-\frac{s^3}{3}} e^{s\xi} ds, \qquad (3.4.4)$$

since $U(s) = Ae^{-\frac{s^3}{3}}$ has no singularities, one can consider the above integral relation as below (let $A = 1$)

$$u(\xi) = \frac{1}{2\pi i}\lim_{L \to +\infty}\int_{-iL}^{iL} e^{-\frac{s^3}{3}} e^{s\xi} ds,$$

making a change of variables $s = iw$ we get

$$u(\xi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iw(\xi + \frac{w^2}{3})} dw = \frac{1}{\pi}\int_{0}^{+\infty} \cos w(\xi + \frac{w^2}{3}) dw, \qquad (3.4.5)$$

because $\sin x$ is an odd and $\cos x$ is an even function. It means that

$$x(\xi)=\frac{1}{\pi}\int_0^{\xi}\left(\int_0^{+\infty}\cos w(\eta+\frac{w^2}{3})dw\right)d\eta=\frac{1}{\pi}\int_0^{+\infty}\left(\int_0^{\xi}\cos w(\eta+\frac{w^2}{3})d\eta\right)dw$$
$$=\frac{1}{\pi}\int_0^{+\infty}\{\sin(w\xi+\frac{w^3}{3})-\sin\frac{w^3}{3}\}\frac{dw}{w},$$

or

$$x(z)=\frac{1}{\pi}\int_0^{+\infty}\{\sin[w\,(\left(\frac{q}{EI}\right)^{\frac{1}{3}}(z-L)+\frac{w^2}{3})]-\sin\frac{w^3}{3}\}\frac{dw}{w}\cdot$$

In addition since there are some kinds of errors in modeling a physical phenomenon, it will be a good idea to model it by a fractional differential equation, therefore the following FDE is suitable for the above mentioned equilibrant rod

$$IE\ D^{2\alpha+1}x=-q(L-z)x',$$

Now let $x'(z)=u(z),\xi=\left(\frac{q}{EI}\right)^{\frac{1}{2\alpha+1}}(z-L)$ to get the following fractional Airy differential equation

$$D^{2\alpha}u(\xi)-\xi u(\xi)=0,$$

the reader could find the above fractional DE under the suitable boundary conditions in the following problem

Problem.3.4.2. Consider the fractional Airy DE

$${}^cD^{2\alpha}u(\xi)+k\,{}^cD^{\beta}u(\xi)+\lambda\xi u(\xi)=0,\quad \frac{1}{2}\le\alpha\le 1,\quad 0\le\beta\le 1,$$

under the conditions

$$u(0)=u'(0)=0.$$

Solution. Taking Laplace transform of the above differential equation we have

$$s^{2\alpha}U(s)+ks^{\beta}U(s)-\lambda U'(s)=0,$$

we will have

$$(s^{2\alpha}+k\,s^{\beta})U(s)=\lambda U'(s),$$

therefore

$$U(s) = A\exp(\frac{s^{2\alpha+1}}{\lambda(2\alpha+1)} + \frac{k}{\lambda}\frac{s^{\beta+1}}{\beta+1}).$$

Use the Titchmarsh theorem to get the following relationship (let $A=1$)

$$u(\xi) = \frac{1}{\pi}\int_0^{+\infty} e^{-r\xi}\{\operatorname{Im} U(re^{-i\pi})\}dr = \frac{1}{\pi}\int_0^{+\infty} e^{-r\xi}\{\operatorname{Im}(e^{\frac{(re^{-i\pi})^{2\alpha+1}}{\lambda(2\alpha+1)} + \frac{k}{\lambda}\frac{(re^{-i\pi})^{\beta+1}}{\beta+1}})\}dr,$$

it means that

$$u(\xi) = \frac{1}{\pi}\int_0^{+\infty} e^{-r\xi}\{\operatorname{Im}(e^{\frac{r^{2\alpha+1}}{\lambda(2\alpha+1)}(-1)^{2\alpha+1} + \frac{k}{\lambda}\frac{r^{\beta+1}}{\beta+1}(-1)^{\beta+1}})\}dr. \qquad (3.4.6)$$

On the other hand we know that

$$(-1)^{2\alpha+1} = e^{(2\alpha+1)\ln(-1)} = e^{i(2\alpha+1)\frac{\pi}{2}}, (-1)^{\beta+1} = e^{(\beta+1)\ln(-1)} = e^{i(\beta+1)\frac{\pi}{2}},$$

substituting in (3.4.6) we get the result as below

$$u(\xi) = \frac{1}{\pi}\int_0^{+\infty} \exp\{-r\xi + \frac{r^{2\alpha+1}}{\lambda(2\alpha+1)}\cos(2\alpha+1)\frac{\pi}{2} + \frac{k}{\lambda}\frac{r^{\beta+1}}{\beta+1}\cos(\beta+1)\frac{\pi}{2}\}\times$$
$$\sin\left\{\frac{r^{2\alpha+1}}{\lambda(2\alpha+1)}\sin(2\alpha+1)\frac{\pi}{2} + \frac{k}{\lambda}\frac{r^{\beta+1}}{\beta+1}\sin(\beta+1)\frac{\pi}{2}\right\}dr.$$

Special case: Let $\lambda = -1, \beta = k = 0$ to find the solution of the fractional problem associated to the elastic rod (mentioned above) as below

$$u(\xi) = \frac{1}{\pi}\int_0^{+\infty} \exp(-r\xi - \frac{r^{2\alpha+1}}{2\alpha+1}\cos(2\alpha+1)\frac{\pi}{2})\times\sin\left\{\frac{r^{2\alpha+1}}{2\alpha+1}\sin(2\alpha+1)\frac{\pi}{2}\right\}dr.$$

And then because $x'(\xi) = u(\xi)$ we have

$$x(\xi) = \frac{1}{\pi}\int_0^{\xi}\left(\int_0^{+\infty} \exp(-r\eta - \frac{r^{2\alpha+1}}{2\alpha+1}\cos(2\alpha+1)\frac{\pi}{2})\times\sin\left\{\frac{r^{2\alpha+1}}{2\alpha+1}\sin(2\alpha+1)\frac{\pi}{2}\right\}dr\right)d\eta,$$

changing the order of integrals we will have

$$x(\xi) = \frac{1}{\pi}\int_0^{+\infty} e^{-\frac{r^{2\alpha+1}}{2\alpha+1}\cos(2\alpha+1)\frac{\pi}{2}} \sin\left\{\frac{r^{2\alpha+1}}{2\alpha+1}\sin(2\alpha+1)\frac{\pi}{2}\right\}(1-e^{-r\xi})\frac{dr}{r},$$

and therefore

$$x(z) = \frac{1}{\pi}\int_0^{+\infty} e^{-\frac{r^{2\alpha+1}}{2\alpha+1}\cos(2\alpha+1)\frac{\pi}{2}} \sin\left\{\frac{r^{2\alpha+1}}{2\alpha+1}\sin(2\alpha+1)\frac{\pi}{2}\right\}(1-e^{-r\left(\frac{q}{EI}\right)^{\frac{1}{2\alpha+1}}(z-L)})\frac{dr}{r}.$$

Problem.3.4.3. Solve the following system of fractional differential equations

$$\begin{cases} D^{\alpha}u(x,t) - D^{\beta}v(x,t) + \dfrac{\partial u}{\partial x}(x,t) = 0 \\ D^{\alpha}v(x,t) + D^{\beta}u(x,t) + \dfrac{\partial v}{\partial x}(x,t) = 0 \end{cases}, \quad 0<\alpha,\beta<1,$$

under the given boundary and initial conditions

$$u(x,0) = v(x,0) = 0, \quad 0<x<1, \quad t>0.$$

Solution. By changing a new variable $w = u + iv$, the above system of equations will be changed to

$$D^{\alpha}w(x,t) + iD^{\beta}w(x,t) = -\frac{\partial w}{\partial x}(x,t),$$

now taking Laplace transform of the above differential equation we get

$$(s^{\alpha} + is^{\beta})W(x,s) = -W_x(x,s),$$

it means that (assume $A = a + ib$)

$$W(x,s) = Ae^{-(s^{\alpha}+is^{\beta})x} = (a+ib)e^{-xs^{\alpha}}(\cos xs^{\beta} + i\sin xs^{\beta})$$

therefore

$$U(x,s) = ae^{-xs^{\alpha}}\cos xs^{\beta} - be^{-xs^{\alpha}}\sin xs^{\beta},$$
$$V(x,s) = ae^{-xs^{\alpha}}\sin xs^{\beta} + be^{-xs^{\alpha}}\cos xs^{\beta},$$

taking inverse Laplace transform we will have

$$u(x,t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(ae^{-xs^{\alpha}}\cos xs^{\beta}-be^{-xs^{\alpha}}\sin xs^{\beta})e^{st}ds,$$
$$v(x,t)=\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}(ae^{-xs^{\alpha}}\sin xs^{\beta}+be^{-xs^{\alpha}}\cos xs^{\beta})e^{st}ds.$$

Special case: Assume that $\alpha=\beta=0.5, u(0,t)=\delta(t), v(0,t)=0$ and therefore $a=1, b=0$ to get the following solutions

$$U(x,s)=e^{-x\sqrt{s}}\cos(x\sqrt{s}),\quad V(x,s)=e^{-x\sqrt{s}}\sin(x\sqrt{s}),$$

to the problem

$$\begin{cases} D^{\frac{1}{2}}u(x,t)-D^{\frac{1}{2}}v(x,t)+\dfrac{\partial u}{\partial x}(x,t)=0 \\ D^{\frac{1}{2}}v(x,t)+D^{\frac{1}{2}}u(x,t)+\dfrac{\partial v}{\partial x}(x,t)=0 \end{cases}.$$

Then by using series expansion of the functions sin and cos we can write

$$U(x,s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}e^{-x\sqrt{s}}(x\sqrt{s})^{2n},\quad V(x,s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}e^{-x\sqrt{s}}(x\sqrt{s})^{2n+1},$$

which can be rewritten as below

$$U(x,s)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\{s^n e^{-x\sqrt{s}}\},\quad V(x,s)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\{\frac{s^{n+1}e^{-x\sqrt{s}}}{\sqrt{s}}\},$$

using the relations

$$\int f(t)\delta^k(t-a)dt=\frac{d^k}{dt^k}f(t)\Big|_{t=a},\quad L^{-1}\{s^k;s->t\}=\delta^{(k)}(t)$$

we have

$$u(x,t)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\{\delta^{(n)}(t)*\frac{xe^{-\frac{x^2}{4t}}}{2t\sqrt{\pi t}}\}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\{\frac{d^n}{d\xi^n}\frac{xe^{-\frac{x^2}{4(t-\xi)}}}{2\sqrt{\pi(t-\xi)^3}}\}_{\xi=0},$$

$$v(x,t)=\sum_{n=0}^{\infty}\frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}\{\delta^{(n+1)}(t)*\frac{e^{-\frac{x^2}{4t}}}{\sqrt{\pi t}}\}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\{\frac{d^{n+1}}{d\xi^{n+1}}\frac{e^{-\frac{x^2}{4(t-\xi)}}}{\sqrt{\pi(t-\xi)}}\}_{\xi=0}$$

Problem.3.4.4. Solve the following partial fractional differential equation

$$aD_t^{\alpha}u(x,t)+bD_t^{\beta}u(x,t)+cu(x,t)+\lambda\frac{\partial u}{\partial x}=0,\quad 0\le x\le 1,$$

under the conditions

$$u(x,0)=0,\quad u(0,t)=t,\quad a,b,\lambda\in R^{+}.$$

Solution. Taking Laplace transform of the above differential equation with respect to t we have

$$\frac{U_x(x,s)}{U(x,s)}=-\lambda(as^{\alpha}+bs^{\beta}+c),$$

now let us solve the above DE of first order to get

$$U(x,s)=A(s)e^{-\lambda x(as^{\alpha}+bs^{\beta}+c)}=A(s)e^{-cx\lambda}(e^{-(\lambda xa)s^{\alpha}})(e^{-(\lambda xb)s^{\beta}})$$

At this point, using boundary condition $u(0,t)=t$ and taking inverse Laplace transform leads to the following relation

$$u(x,t)=e^{-cx\lambda}\{L^{-1}(\frac{e^{-(\lambda xa)s^{\alpha}}}{s})*L^{-1}(\frac{e^{-(\lambda xb)s^{\beta}}}{s})\},$$

it suffices to evaluate the inverse Laplace transform of the above function considering problem 2.8 and then using Laplace transform of convolution of functions

$$u(x,t)=e^{-cx\lambda}\{\frac{1}{\pi}\int_0^\infty \frac{1}{\eta}e^{-\lambda xa\eta^\alpha\cos\alpha\pi-t\eta}\sin(\lambda xa\eta^\alpha\sin\alpha\pi)d\eta\,*$$

$$\frac{1}{\pi}\int_0^\infty \frac{1}{\xi}e^{-\lambda xb\xi^\beta\cos\beta\pi-t\xi}\sin(\lambda xb\xi^\beta\sin\beta\pi)d\xi\}$$

$$=\frac{e^{-cx\lambda}}{\pi^2}\int_0^t\left(\int_0^\infty \frac{1}{\eta}e^{-\lambda xa\eta^\alpha\cos\alpha\pi-u\eta}\sin(\lambda xa\eta^\alpha\sin\alpha\pi)d\eta\right)\times$$

$$\left(\int_0^\infty \frac{1}{\xi}e^{-\lambda xb\xi^\beta\cos\beta\pi-(t-u)\xi}\sin(\lambda xb\xi^\beta\sin\beta\pi)d\xi\right)du.$$

Changing the order of integrals one has

$$u(x,t)=\frac{e^{-cx\lambda}}{\pi^2}\int_0^\infty \frac{1}{\eta}e^{-\lambda xa\eta^\alpha\cos\alpha\pi}\sin(\lambda xa\eta^\alpha\sin\alpha\pi)\times$$

$$\left(\int_0^\infty \frac{1}{\xi}e^{-\lambda xb\xi^\beta\cos\beta\pi}\sin(\lambda xb\xi^\beta\sin\beta\pi)\frac{e^{-t\eta}-e^{-t\xi}}{\xi-\eta}d\xi\right)d\eta.$$

Problem.3.4.5. Consider fractional – order system of differential equations

$$\begin{bmatrix} {}_aD^\alpha & {}_bD^\beta \\ {}_{a'}D^\alpha & {}_{b'}D^\beta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} f(t) \\ g(t) \end{bmatrix}.$$

Solution. One can rewrite the above system as below

$$\begin{bmatrix} {}_aD^\alpha-c_1 & {}_bD^\beta-c_2 \\ {}_{a'}D^\alpha-c_3 & {}_{b'}D^\beta-c_4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} f(t) \\ g(t) \end{bmatrix},$$

taking Laplace transform we will have

$$\begin{cases} (s^\alpha-c_1)X(s)+(s^\beta-c_2)Y(s)=F(s) \\ (s^\alpha-c_3)X(s)+(s^\beta-c_4)Y(s)=G(s) \end{cases},$$

then using Cramer's rule we get

$$X(s)=\frac{s^{\beta}(F(s)-G(s))-F(s)c_4+G(s)c_2}{s^{\alpha}(c_2-c_4)+s^{\beta}(c_3-c_1)+(c_1c_4-c_2c_3)},$$
$$Y(s)=\frac{s^{\alpha}(G(s)-F(s))+F(s)c_3-G(s)c_1}{s^{\alpha}(c_2-c_4)+s^{\beta}(c_3-c_1)+(c_1c_4-c_2c_3)}\cdot$$

At this point, using invesion formula for Laplace transform to obtain

$$x(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{s^{\beta}(F(s)-G(s))-F(s)c_4+G(s)c_2}{s^{\alpha}(c_2-c_4)+s^{\beta}(c_3-c_1)+(c_1c_4-c_2c_3)}\;e^{st}ds,$$
$$y(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{s^{\alpha}(G(s)-F(s))+F(s)c_3-G(s)c_1}{s^{\alpha}(c_2-c_4)+s^{\beta}(c_3-c_1)+(c_1c_4-c_2c_3)}\,e^{st}ds\cdot$$

Problem.3.4.6. Solve the following system of fractional differential equations

$$\begin{cases}{}_0^C D_t^{\alpha}x+{}^C D_t^{\beta}y=-x-e^{-t}\\ {}_0^C D_t^{\alpha}x+2\,{}^C D_t^{\beta}y=-2x-2y\end{cases};\quad x(0)=-1,y(0)=1.$$

Proof. Taking Laplace transform of the both equations of the system we have

$$\begin{cases}(s^{\alpha}+1)X(s)+s^{\beta}Y(s)=-\dfrac{1}{s+1}\\ (s^{\alpha}+2)X(s)+2(s^{\beta}+1)Y(s)=1\end{cases},$$

which can be solved using Cramer’s rule as below

$$X(s)=\frac{\begin{vmatrix}-\dfrac{1}{s+1} & s^{\beta}\\ 1 & 2(s^{\beta}+1)\end{vmatrix}}{\begin{vmatrix}s^{\alpha}+1 & s^{\beta}\\ s^{\alpha}+2 & 2(s^{\alpha}+1)\end{vmatrix}}=\frac{3s^{\beta}-s^{\beta+1}+2}{s^{\alpha+\beta}+s^{\alpha}+1},$$

and in a similar way

$$Y(s)=\frac{1-3(s+1)-s^{\alpha}(s+1)-s^{\alpha}}{(s+1)(s^{2\alpha}+s^{\alpha}+2)}\cdot$$

Let $\alpha = \beta = \dfrac{1}{2}$ then we have

$$X(s) = \frac{3\sqrt{s} - s\sqrt{s} + 2}{s + \sqrt{s} + 1}, \quad Y(s) = \frac{1 - 3(s+1) - \sqrt{s}(s+1) - \sqrt{s}}{(s+1)(s + \sqrt{s} + 2)}.$$

We can assume the above relationships as below

$$X(s) = F(\sqrt{s}) \to F(s) = \frac{3s - s^2 + 2}{s^2 + s + 1},$$
$$Y(s) = G(\sqrt{s}) \to G(s) = \frac{1 - 3(s^2+1) - s(s^2+1) - s}{(s^2+1)(s^2 + s + 2)}.$$

Now by manipulating the above functions and taking inverse Laplace transform of them, we will have

$$f(t) = L^{-1}\{F(s)\} = \frac{2}{\sqrt{3}} e^{-\frac{t}{2}} \left(\sin(\frac{\sqrt{3}}{2} t) + 2\sqrt{3}\cos(\frac{\sqrt{3}}{2} t) \right) - \delta(t),$$
$$g(t) = L^{-1}\{G(s)\} = -\frac{4}{\sqrt{7}} e^{-\frac{t}{2}} \sin(\frac{\sqrt{7}}{2} t) - \cos t.$$

Now by using Efros theorem (see [6]) we have

$$x(t) = \frac{1}{2t\sqrt{\pi t}} \int_0^\infty e^{-\frac{\tau^3}{4t}} \tau \left\{ \frac{2}{\sqrt{3}} e^{-\frac{\tau}{2}} \left(\sin(\frac{\sqrt{3}}{2}\tau) + 2\sqrt{3}\cos(\frac{\sqrt{3}}{2}\tau) \right) - \delta(\tau) \right\} d\tau,$$
$$y(t) = -\frac{1}{2t\sqrt{\pi t}} \int_0^\infty e^{-\frac{\tau^3}{4t}} \tau \left\{ \frac{4}{\sqrt{7}} e^{-\frac{\tau}{2}} \sin(\frac{\sqrt{7}}{2}\tau) + \cos\tau \right\} d\tau.$$

4 Conclusion

In this work, the Fourier, Laplace and L_2 transforms are implemented to solve boundary value problems of fractional order. The paper is devoted to study the applications of various integral transforms to solve different types of Airy differential equations of integer and fractional order. They conclude by remarking that many identities involving various integral transform can be obtained and some other definite integrals can be evaluated by applying the results considered here. It may be concluded that the transform method is very efficient tool in finding exact solutions for ordinary differential equations and systems of differential equations of fractional orders. Finally, illustrative examples for the above mentioned transforms are provided.

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Competing Interests

Authors have declared that no competing interests exist.

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