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Some New Exact Traveling Wave Solutions for the Zhiber-Shabat Equation

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Abstract

In this paper, we implemented a very important method to solve nonlinear partial differential equations known as the $\exp(-\phi(\xi))$-expansion method. Recently, there are several methods being constructed for finding analytical solutions of nonlinear partial differential equations. However, the $\exp(-\phi(\xi))$-expansion method is more effective and useful for solving the nonlinear evolution equations. With the help of this method, we are investigated the exact traveling wave solutions of the Zhiber-Shabat equation. The obtaining exact solutions of this equation are describe many physical phenomena in mathematical physics such as solid state physics, plasma physics, nonlinear optics, chemical kinetics and quantum field theory. Further, three-dimensional plots of the solutions such as solitons, cuspon, periodic, singular kink and bell type are also given to visualize the dynamics of the equation.

Keywords: The $\exp(-\phi(\xi))$-expansion method, the Zhiber-Shabat equation, nonlinear evolution equations, traveling wave solutions, solitary wave solutions.
Mathematics Subject Classification: 35K99, 35P05, 35P99.

1 Introduction

Nonlinear partial differential equations (NLPDEs) have been widely applied in many branches of applied sciences such as fluids dynamics, bio-mechanics, chemical physics, particle physics, quantum field theory, optical fibers and plasma physics etc. In the research of the theory of NLPDEs, searching for more explicit exact solutions to NLPDEs is one of the most fundamental and significant studies in recent years. With the help of computerized symbolic computation, much work has focused on the various extensions and applications of the known algebraic methods to construct the solutions to NLPDEs. There have been a various types of powerful methods for solving NLPDEs. Recently, many types of effective methods have been proposed to calculate exact solutions of nonlinear NLPDEs. For example, the improved F-expansion method [1], the Jacobi elliptic function expansion method [2, 3], the projective Riccati equation method [4], the tanh-function method [5-8], the inverse scattering transform method [9], the Exp-function

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method [10-13], the tanh-method [14,15], the extended tanh-method [16, 17, 18], the exponential function method[18], the sec^p-tanh^p method [18], the $(G'/G)-$expansion method [19,20], the homogeneous balance method [21-23], sine-cosine method [24,25] and the $\exp(-\varphi(\xi))$ -expansion method [26,27] etc. But nobody has researched the application of the $\exp(-\varphi(\xi))$-expansion method to construct the exact travelling wave solutions of the Zhiber-Shabat equation, which is very important to describe many physical phenomena in mathematical physics such as solid state physics, plasma physics, nonlinear optics, chemical kinetics and quantum field theory.

The paper is organized as follows: In section 2, we have presented the briefly description of the $\exp(-\Phi(\xi))$ expansion method. Section 3 is devoted to derive the travelling wave solutions of the Zhiber-Shabat equation using this method. The physical interpretation and graphical representations of the solutions are presented in section 4. The conclusion is given by in section 5.

2 Description of the $\exp(-\Phi(\xi))$ -Expansion Method

Let us consider a general nonlinear PDE in the form

$$F(u,u_t,u_x,u_{xx},u_{tt},u_{tx},\ldots), \tag{1}$$

where $u=u(x,t)$ is an unknown function, F is a polynomial in $u(x,t)$ and its derivatives in which highest order derivatives and nonlinear terms are involved and the subscripts stand for the partial derivatives. The main steps of this method are as follows:

Step 1: Combine the real variables x and t by a complex variable ξ

$$u(x,t)=u(\xi),\ \xi=x\pm ct, \tag{2}$$

where V is the speed of the traveling wave. The traveling wave transformation (2) converts Eq. (1) into an ordinary differential equation (ODE) for $u=u(\xi)$:

$$\Re(u,u',u'',u''',\cdots), \tag{3}$$

where $\Re$ is a polynomial of u and its derivatives and the superscripts indicate the ordinary derivatives with respect to ξ.

Step 2. Suppose the traveling wave solution of Eq. (3) can be expressed as follows:

$$u(\xi)=\sum_{i=0}^{N}A_i(\exp(-\Phi(\xi)))^{i}, \tag{4}$$

where A_i $(0\le i\le N)$ are constants to be determined, such that $A_N\neq 0$ and $\Phi=\Phi(\xi)$ satisfies the following ordinary differential equation:

$$\Phi'(\xi) = \exp(-\Phi(\xi)) + \mu \exp(\Phi(\xi)) + \lambda, \tag{5}$$

Eq. (5) gives the following solutions:

Family 1: When $\mu \neq 0,\ \lambda^2 - 4\mu > 0,$

$$\Phi(\xi) = \ln\left(\frac{-\sqrt{(\lambda^2 - 4\mu)} \tanh\left(\frac{\sqrt{(\lambda^2 - 4\mu)}}{2} (\xi + E) \right) - \lambda}{2\mu} \right) \tag{6}$$

Family 2: When $\mu \neq 0,\ \lambda^2 - 4\mu < 0,$

$$\Phi(\xi) = \ln\left(\frac{\sqrt{(4\mu - \lambda^2)} \tan\left(\frac{\sqrt{(4\mu - \lambda^2)}}{2} (\xi + E) \right) - \lambda}{2\mu} \right) \tag{7}$$

Family 3: When $\mu = 0,\ \lambda \neq 0,$ and $\lambda^2 - 4\mu > 0,$

$$\Phi(\xi) = -\ln\left(\frac{\lambda}{\exp(\lambda(\xi + E)) - 1} \right) \tag{8}$$

Family 4: When $\mu \neq 0,\ \lambda \neq 0,$ and $\lambda^2 - 4\mu = 0,$

$$\Phi(\xi) = \ln\left(-\frac{2(\lambda(\xi + E) + 2)}{\lambda^2 (\xi + E)} \right) \tag{9}$$

Family 5: When $\mu = 0,\ \lambda = 0,$ and $\lambda^2 - 4\mu = 0,$

$$\Phi(\xi) = \ln(\xi + E) \tag{10}$$

$A_N, \cdots\cdots, c, \lambda, \mu$ are constants to be determined latter, $A_N \neq 0,$ the positive integer N can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (3).

Step 3: Substitute Eq. (4) into Eq. (3) and then we account the function $\exp(-\Phi(\xi))$. As a result of this substitution, we get a polynomial of $\exp(-\Phi(\xi))$. We equate all the coefficients of same power of $\exp(-\Phi(\xi))$ are equal to zero which yielding a system of algebraic equations. Solving the obtaining system, we can find the value of $A_1, A_2 \cdots\cdots, c$. Substituting the values of $A_1, A_2 \cdots\cdots, c$ into Eq. (4) along with general solutions of Eq. (5) completes the determination of the solution of Eq. (1).

3 Some new Exact Travelling Wave Solutions of Zhiber-Shabat Equation

Let us consider Zhiber-Shabat equation [28] of the following form,

$$\frac{\partial^2 Z}{\partial x \partial t} + pe^{Z} + q e^{-Z} + r e^{-2Z} = 0 \tag{11}$$

where, p, q and r are arbitrary constant. For $q = 0$, Eq. (11) reduces to the Dodd-Bullough-Mikhailov equation. For $p = 0,\ q = -1,\ r = 1$, Eq. (11) gives the Tzitzeica-Dodd-Bullough equation. . For $r = 0$, we obtain the sinh-Gordon equation. The above mention equations arise in many applications in mathematical physics.

If we introduce the following transformations

$$u(x,t) = e^{Z(x,t)} = u(\xi), \quad \xi = x - ct \tag{12}$$

then the Eq. (14) can be reducing to a nonlinear ordinary differential equation as follows:

$$-c(uu'' - u'^2) + pu^3 + qu + r = 0 \tag{13}$$

where, primes denotes the differentiation with regard to ξ .By balancing uu'' and u^3, the pole of the equation (13) is $N = 2$. Therefore, the $\exp(-\phi(\xi))$ -expansion method admits the solution of (13) in the form

$$u(\xi) = A_0 + A_1 \exp(-\phi(\xi)) + A_2 (\exp(-\phi(\xi)))^2, \ \ A_2 \neq 0 \tag{14}$$

By Substituting (5) and (14) into the Eq. (13) and equating the coefficient of $(\exp(-\phi(\xi)))^i$, $(i = 0,1,2,\cdots,6)$ are equal to zero, yielding a set of algebraic equations as follows:

$$-2cA_2^2 + pA_2^3 = 0 \tag{15}$$

$$3pA_1A_2^2 - 2cA_2^2\lambda - 4cA_1A_2 = 0 \tag{16}$$

$$-6cA_0A_2 - cA_1^2 + 3pA_0A_2^2 - 5cA_1A_2\lambda + 3pA_1^2A_2 = 0 \tag{17}$$

$$pA_1^3 - 10cA_0A_2\lambda - 2cA_1A_2\mu - cA_1^2\lambda + 6pA_0A_1A_2 - cA_1A_2\lambda^2 + 2cA_2^2\mu\lambda - 2cA_0A_1 = 0 \tag{18}$$

$$2cA_2^2\mu^2 + 3pA_0A_1^2 - 8cA_0A_2\mu - 3cA_0A_1\lambda - 4cA_0A_2\lambda^2 + qA_2 + 3pA_0^2A_2 + cA_1A_2\mu\lambda = 0 \tag{19}$$

$$2cA_1A_2\mu^2 + 3pA_0^2A_1 - 2cA_0A_1\mu - cA_0A_1\lambda^2 + qA_1 + cA_1^2\lambda\mu - 6cA_0A_2\mu\lambda = 0 \tag{20}$$

$$r - 2cA_0A_2\mu^2 + cA_1^2\mu^2 + qA_0 - cA_0A_1\lambda\mu + pA_0^3 = 0 \tag{21}$$

Solving the algebraic Eq. (15) to (21), we obtain a set of solution as follows:

$$\Big\{ c = c,\quad p = \frac{2c}{A_2}, r = \frac{c(4A_0A_2^2\mu^2 - A_2^3\lambda^2\mu^2 - A_0^2A_2\lambda^2 - 8A_0^2A_2\mu + 4A_0^3 + 2A_0A_2^2\lambda^2\mu)}{A_2},$$
$$q = -\frac{c(2A_2^2\mu^2 - A_0A_2\lambda^2 - 8A_0A_2\mu + 6A_0^2 + A_2^2\lambda^2\mu)}{A_2}, A_0 = A_0,\quad A_1 = A_2\lambda,\quad A_2 = A_2 \Big\} \tag{22}$$

where c, λ, A_0, A_2 and μ are arbitrary constants.
By substituting (22) into Eq. (17), we have

$$u(\xi) = A_0 + A_2\lambda\exp(-\phi(\xi)) + A_2(\exp(-\phi(\xi)))^2 \tag{23}$$

where $\xi = x - ct.$

Again, by use of the Eq. (6), (7), (8), (9), (10), (12) and (23), the travelling wave solutions of the equation Eq. (11) are obtained as follows:
When $\mu \neq 0,\ \lambda^2 - 4\mu > 0,$ we find that

$$Z_1(x,t) = \ln\left\{ A_0 - \lambda A_2\left(\frac{2\mu}{\sqrt{\Omega}\tanh(\frac{\sqrt{\Omega}}{2}(x - ct + E)) + \lambda}\right) + A_2\left(\frac{2\mu}{\sqrt{\Omega}\tanh(\frac{\sqrt{\Omega}}{2}(x - ct + E)) + \lambda}\right)^2 \right\}, \tag{24}$$

where $\Omega = \lambda^2 - 4\mu$, and E is arbitrary constant.
When $\mu \neq 0,\ \lambda^2 - 4\mu < 0,$ we find that

$$Z_2(x,t) = \ln\left\{A_0 + \lambda A_2\left(\frac{2\mu}{\sqrt{\Omega}\tan(\frac{\sqrt{\Omega}}{2}(x-ct+E)) - \lambda}\right) + A_2\left(\frac{2\mu}{\sqrt{\Omega}\tan(\frac{\sqrt{\Omega}}{2}(x-ct+E)) - \lambda}\right)^2\right\}, \quad (25)$$

where $\Omega = 4\mu - \lambda^2$, and E is arbitrary constant.

When $\mu = 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu > 0$, we find that

$$Z_3(x,t) = \ln\left\{A_0 + \lambda A_2\left(\frac{\lambda}{\exp(\lambda(x-ct+E))-1}\right) + A_2\left(\frac{\lambda}{\exp(\lambda(x-ct+E))-1}\right)^2\right\}, \quad (26)$$

where E is arbitrary constant.

When $\lambda^2 - 4\mu = 0, \mu \neq 0, \lambda \neq 0$, we find that

$$Z_4(x,t) = \ln\left\{A_0 - \lambda A_2\left(\frac{\lambda^2(x-ct+E)}{\lambda(x-ct+E)+2}\right) + A_2\left(\frac{\lambda^2(x-ct+E)}{\lambda(x-ct+E)+2}\right)^2\right\}, \quad (27)$$

where E is arbitrary constant.

When $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0$, we find that

$$Z_5(x,t) = \ln\left\{A_0 - \lambda A_2\left(\frac{1}{x-ct+E}\right) + A_2\left(\frac{1}{x-ct+E}\right)^2\right\}, \quad (28)$$

where E is arbitrary constant.

4 Physical Interpretation

In this section, we describe the physical interpretation and graphical representation of the solutions of the Zhiber-Shabat equation.

4.1 Interpretations

Solitons are solitary waves with elastic scattering property, which described many physical phenomena in soliton physics. Soliton retain their shapes and speed after colliding with each other. Soliton solutions also give rise to particle-like structures, such as magnetic monopoles etc. So, soliton are everywhere in the nature. The solution $Z_1(x,t)$ in Fig. 1 of the equation (11) is represented the soliton solution for $E=1$, $\mu=-1$, $A_0=1$, $A_2=2$, $c=5$, $\lambda=1$ with $-10 \le x,t \le 10$.

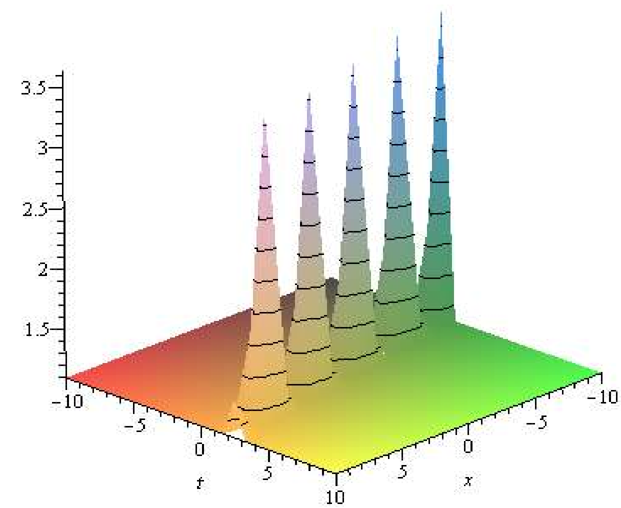

Fig. 1. Exact soliton solution, shape of solution $Z_1(x,t)$ **when** $E=1,\ \mu=-1,\ A_0=1,$ $A_2=2,\ c=5,\ \lambda=1$ **and** $-10\le x,t\le 10$

Solution $Z_3(x,t)$ of the equation (11) is cuspon which is shown in Fig. 2 for $E=1,\ \mu=0,$ $A_0=2,\ A_2=3,\ c=1,\ \lambda=0.5$ with $-10\le x,t\le 10$ .Cuspons are other kinds of solitons where solution exhibits cusps at their crests.

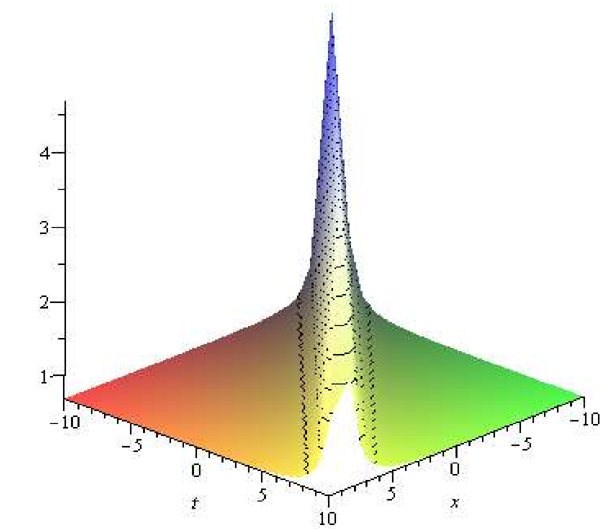

Fig. 2. Exact cuspon solution, shape of solution $Z_3(x,t)$ **when** $E=1,\ \mu=0,\ A_0=2,\ A_2=3,$ $c=1,\ \lambda=0.5$ **and** $-10\le x,t\le 10$

The solution $Z_2(x,t)$ of the equation (11) is presented the periodic travelling wave solution for various values of the physical parameters. The Fig. 3 has been shown the shape of the solution $Z_2(x,t)$ for $E=1,\ \mu=1,\ A_0=1,\ A_2=2,\ c=-1,\ \lambda=-1$ with $-10 \le x,t \le 10$.

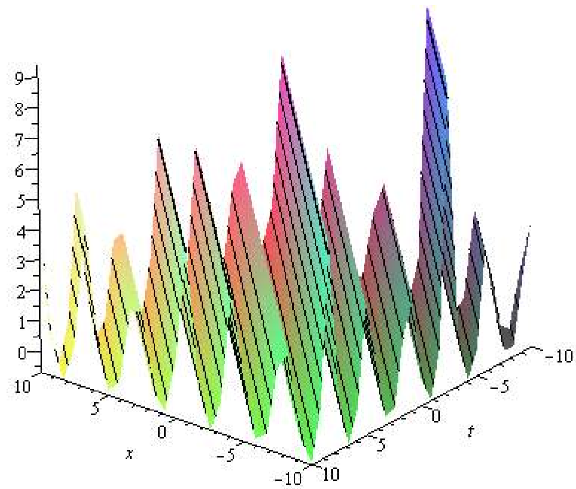

Fig. 3. Exact periodic travelling wave solution, shape of solution $Z_2(x,t)$ when $E=1$, $\mu=1,\ A_0=1,\ A_2=2,\ c=-1,\ \lambda=-1$ and $-10 \le x,t \le 10$

The solution $Z_4(x,t)$ of the equation (11) is described the exact singular kink type solution which is shown in Fig. 4 for $E=1,\ \mu=1,\ A_0=1.5,\ A_2=-2,\ c=-1,\ \lambda=2$ with $-10 \le x,t \le 10$.

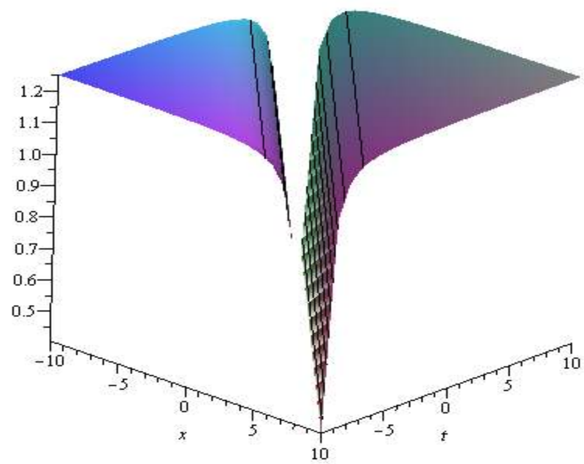

Fig. 4. Exact singular kink type travelling wave solution, shape of solution $Z_4(x,t)$ when $E=1,\ \mu=1,\ A_0=1.5,\ A_2=-2,\ c=-1,\ \lambda=2$ and $-10 \le x,t \le 10$

Finally, solution $Z_5(x,t)$ of the equation (11) is represented the exact Bell type solitary wave solution which is shown in Fig. 5 for $c=-0.5,\ A_0=1,\ A_1=1.5,\ E=1,\ \mu=0,\ \lambda=0$ with $-10\le x,t\le 10$.

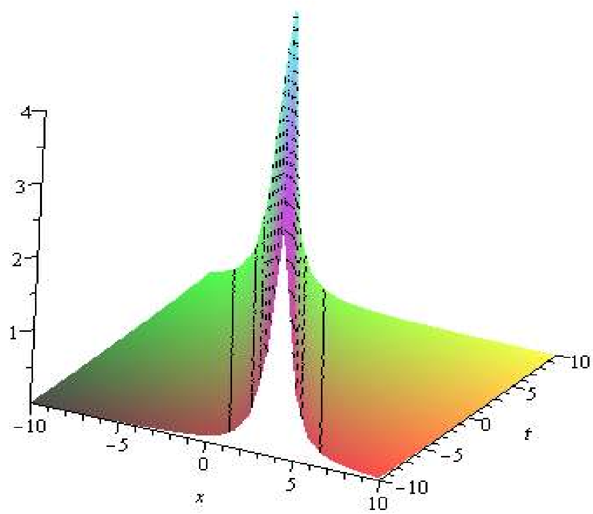

Fig. 5. Bell type solitary wave solution, shape of solution $Z_5(x,t)$ when $c=-0.5,\ A_0=1,\ A_1=1.5,\ E=1,\ \mu=0,\ \lambda=0$ and $-10\le x,t\le 10$

4.2 Graphical Representations

The graphical illustrations of the solutions are given below in the figures (Figs. 1-5) with the aid of Maple.

5 Conclusion

In this paper, the $\exp(-\phi(\xi))$-expansion method has been successfully applied to construct new travelling wave solutions for the Zhiber-Shabat equation. The performance of this method is reliable, convincing and can be used to other NLEEs in finding exact solutions. A variety of distinct physical structures such as solitons, cuspon, periodic, singular kink and bell type solitary wave solutions were formally derived. Although the method has a lot of merit it has a few drawbacks, such as, sometimes the method gives solutions in disguised versions of known solutions that may be found by other methods.

Competing Interests

Authors have declared that no competing interests exist.

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