



A New Definition of Limit of Periodic Function and Periodic g-Contractive Mapping at Infinity

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Abstract

Limit is a basic concept of calculus. However, according to the updated definition, the limit of periodic function at infinity is not in existence. This conclusion of description does not suit with the periodic phenomenon. For example, the temperature on earth is changed periodically every year since the birth of the earth (viewed as $t = 0$). Today (viewed as $t \rightarrow \infty$) the temperature on earth is continuing. Continuation means that the limit exists. In this paper, a new definition of limit of periodic function and periodic g-contractive mapping at infinity is defined by the value of its initial point based on transformation of variables. Similar definition is made for g-contractive ratio of periodic g-contractive mapping with k-related fixed points. These definitions can be used to describe the k-polar problems and calculation the limit of combinations of periodic functions at infinity. Furthermore, the new definition on contractive ratio of periodic iterative g-contractive mapping at infinity can help us to find the constant G and improves the application of the periodic iterative g-contractive mapping theorem.

Keywords: Limit; periodic function; fixed point theorem; Banach contractive mapping theorem; g-contractive mapping theorem; periodic g-contractive mapping theorem.

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1 Introduction

Limit is a fundamental important base of concepts: continuity; derivative; differentiability; integrability; etc. And those concepts form the foundation of developing of calculus. The definition of limit of a function $f(t)$ at t_0 in updated text books, which was originally formed by Cauchy et al. in 1821 [1], is:

$$\lim_{t \rightarrow t_0} f(t) = C. \quad (1-1)$$

Where C is a constant, the limit of $f(t)$ at t_0 . If for all $t \in [a, b]$ $f(t)$ has limit, then $f(t)$ is called to have limit in $[a, b]$. According to the definition of limit of a periodic function $f(t) = f(t + k)$, $t \in (-\infty, \infty)$, $k = \text{const}$.

$$\lim_{t \rightarrow \infty} f(t) = ? \quad (1-2)$$

The answer of (1-2) in any updated text book is: $f(t)$ has no limit at $t \rightarrow \infty$ (see [2]).

The limit of periodic function has no limit ($t \rightarrow \infty$), or only has one limit ($t \rightarrow t_0$) if it exists, do not consist with periodic phenomenon. Periodic phenomenon is an universal phenomenon, which existed in nature and bio-circle (including human society) long-long ago. For examples, 'in earth science, periodic alternations between cold and warm eras occur. In astronomy, the knowledge of periodic alternative in solar activity and the positions of stars are used. The ocean coast is attacked periodically by tides.' [3]. In bio-circle, some birds, animals migrate seasonally. In human society, regulations, habits etc. are set up with the nature periodically. All these examples show that the limit(s) is/are existing for $t \rightarrow \infty$ (the starting of the earth-birth is viewed as $t = 0$, today it is viewed as $t \rightarrow \infty$). Thus, the definition of limit needs to be modified.

1.1 Periodic Phenomena with k-polar

In nature, periodic phenomenon exists in k-polar (extreme points). For examples, in bio-circle, two-polar of "egg and bird", three-polar of "egg, polliwog, and frog", etc is well known. To describe the phenomenon, the limit of periodic function or mapping should have k-related fixed points.

As an important role in analysis, fixed point theorem is one of the active mathematical branch. A lot of researches on contractive mapping have been done [4]. Among these works, the Banach contraction mapping theorem [5] is the most important and widely used theorem. However, the Banach contraction mapping theorem needs the contractive ratio less than a constant lesser than 1. While, the g-contractive mapping theorem [6] allows some of contractive ratios equal to or greater than 1, if the geometric mean of the contractive ratios is less than a constant lesser than 1. Further more, the g-contraction mapping theorem suits for periodic mapping with k-related fixed points.

The idea of g-contraction mapping was originally proposed by the author in 1983 [6]. In it, the author first pointed out the restriction of Banach fixed point theorem and then proved a more general g-contractive mapping theorem with lesser restriction and has a unique set of k related fixed points suited for describing periodic phenomenon. The concept of g-contractive mapping issued by an iteration method with the quickest convergence [7], which had been used by others [8] and it showed that "this method converges quicker than the Lamweber's method, and agrees with experimental data well". In it, different rules (functions or mappings) are used for different steps of the iteration process, so as to obtain the quickest convergence, while in Banach's iteration only one mapping is used in all steps of iteration process.

The g-contraction mapping theorems have applications in the fields of mechanics [9], prey-predator system, stock price [10,11,12], and the analysis of equilibrium state [13]. In this paper, the limit of g-contractive ratio at infinity is defined similarly, and some related articles [14-17] are mentioned in references.

In section 2, a new definition of periodic function at infinity is defined which is based on the transformation of variables, and it applies to “egg comes first or chicken comes first” problem.

In section 3, the limit of combination of periodic functions at infinity is studied with examples.

In section 4, the limit of g-contractive ratio of periodic g-contractive mapping at infinity is similarly defined. Finally, a conclusion is provided.

2 The Value of Initial Point is defined as the Limit of Periodic Function at Infinity

Definition 2.1

$$f(\tau_0) \equiv \lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} f(t + nk), \quad (2-1)$$

Proof:

Make a transformation of variables,

$$t = \tau_0 + nk, \quad t \in (-\infty, \infty), \quad \tau_0 \in [-k, k], \quad n = 0, 1, 2, \dots, \quad (2-2)$$

Where $\tau_0 \in [-k, k]$ is an initial point. On both sides of (2-2) one to one correspondence holds.

$$\lim_{t \rightarrow \infty} f(t) = \lim_{n \rightarrow \infty} f(\tau_0 + nk) = f(\tau_0), \quad (2-3)$$

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2.1 Application to “Egg Comes First or Chicken Comes First” Problem

The “egg comes first or chicken comes first” problem was frequently asked in our childhood with no answer given by adults. The answer of such a problem of cycling path depends on the position setting of a referred point (starting point, initial point) like τ_0 in (2-2).

That is: If the referred point is put on the time before the laying of an egg, then chicken exists first; if the referred point is put on the time before the chicken is incubated from the egg, then, the egg exists first.

3 The Calculation of Limit of Combination of Periodic Functions at Infinity

In section 2, periodic function has limit $f(\tau_0)$ at $t \rightarrow \infty$. Thus, the combination of periodic functions also has limit.

$$3.1 \quad f(t + nk) = f_1(t + nk_1) \cap f_2(t + nk_2)$$

$$\begin{aligned} \lim_{t \rightarrow \infty} f(t) &= \lim_{t \rightarrow \infty} f_1(t) \cap \lim_{t \rightarrow \infty} f_2(t) \\ &= f_1(\tau_{01}) \cap f_2(\tau_{02}) = f(\tau_0), \quad k = k_1 + k_2, \end{aligned} \quad (3-1)$$

Example 3-1. Find the limit of series d, b, d, b,... formed by statement “to hold a dinner party every 3 days after a banquet, and to hold a banquet every 5 days after a dinner party”(selected from <Romance of Three Kingdoms> -- a famous history story of China). The mathematical expression of the statement is:

$$\begin{aligned} 1. f_d(t + nk) &= d, t \in [4, (4+6)n], \\ &= 0, t \notin [4, (4+6)n], \end{aligned} \tag{3-2}$$

$$\begin{aligned} 2. f_b(t + nk) &= b, t \in [10, (4+6)n] \\ &= 0, t \notin [10, (4+6)n], \end{aligned} \tag{3-3}$$

$$f(t + nk) = f_d(t + nk) \cap f_b(t + nk), \tag{3-4}$$

$$\lim_{t \rightarrow \infty} f(t) = \{d, t \in [4, 10n]\} \cap \{b, t \in [10, 10n]\}, n = 0, 1, 2, \dots, \tag{3-5}$$

The series is: 0,0,0,d,0,0,0,0,0,b,0,0,0,d,0,...

$$3.2 f(t + nk) = f_1(t + nk_1) \cup f_2(t + nk_2).$$

$$\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} f_1(t) \cup \lim_{t \rightarrow \infty} f_2(t) = f_1(\tau_{01}) \cup f_2(\tau_{02}) = f(\tau_0), \tag{3-6}$$

Example 3-2. Mathematical express of cycling four sessions very year.

Solution. For some warmer cities, we have

$$\begin{aligned} 1. f_{Spr}(t) &= f_{Spr}(t + 12n) = Spr, t \in [Feb - Apr] \\ &= 0, t \notin [Feb - Apr], \end{aligned} \tag{3-7}$$

$$\begin{aligned} 2. f_{Sum}(t) &= f_{Sum}(t + 12n) = Sum, t \in [May - Aug] \\ &= 0, t \notin [May - Aug], \end{aligned} \tag{3-8}$$

$$\begin{aligned} 3. f_{Aut}(t) &= f_{Aut}(t + 12n) = Aut, t \in [Sep - Nov] \\ &= 0, t \notin [Sep - Nov], \end{aligned} \tag{3-9}$$

$$\begin{aligned} 4. f_{Win}(t) &= f_{Win}(t + 12n) = Win, t \in [Dec - Jan] \\ &= 0, t \notin [Dec - Jan], \end{aligned} \tag{3-10}$$

$$f(t) = f_{Spr}(t) \cup f_{Sum}(t) \cup f_{Aut}(t) \cup f_{Win}(t)$$

$$\lim_{t \rightarrow \infty} f(t) = \{Win, Spr, Spr, Spr, Sum, Sum, Sum, Sum, Aut, Aut, Aut, Win\} \tag{3-11}$$

$$3.3 f_1(t) \cup [f_2(t) \cap f_3(t)] = [f_1(t) \cup f_2(t)] \cap [f_1(t) \cup f_3(t)], \tag{3-12}$$

$$f_1(t) \cap [f_2(t) \cup f_3(t)] = [f_1(t) \cap f_2(t)] \cup [f_1(t) \cap f_3(t)], \tag{3-13}$$

4 Limit of Contractive Ratio s_i of Periodic g -contractive Mapping at Finite

An iteration series

$$x_{i+1} = T_i x_i = T_i \circ T_{i-1} \circ \dots \circ T_0 x_0, \quad (4-1)$$

Where T_i denotes g -contractive mapping on X , $x_i \in X$, $i = 0, 1, 2, \dots$. X represents a specified space, e.g. a Banach space, a Hilbert space, with corresponding distance $\|x - y\|$ between x and y .

$f \circ g = f[g(\cdot)]$ means the composition of f and g . The contractive ratio

$$r_j = \frac{\|x_{j+2} - x_{j+1}\|}{\|x_{j+1} - x_j\|}, j = 1, 2, 3, \dots, \quad (4-2)$$

The geometric mean of contractive ratio (g -contractive ratio)

$$s_n = (r_n r_{n-1} \dots r_1)^{1/n}, n = 1, 2, 3, \dots, \quad (4-3)$$

$$s_n \leq G \text{ (constant)} < 1, \quad (4-4)$$

Definition 4.1 A sequential composite mapping $\{T_j\}$ is called “the periodic mapping with period k ”, if

$$T_j = T_{j+nk} = P_{j+1}^n, (j = 1, 2, \dots, k, n \in \mathbb{N} = 1, 2, \dots). \quad (4-5)$$

And is denoted by P_{j+1}^n , where the superscript n denotes the times of cycling,

The subscript $j \in K = (1, 2, \dots, k)$ indicates the mapping at the corresponding space

X_j , all X_j are included in X . i.e.,

$$T_j: X_j \rightarrow X_{j+1}, \text{ or} \\ x_{j+1} = T_j x_j = T_j \circ T_{j-1} \circ \dots \circ T_{j-k+1} x_{j+1-k}, \quad (4-6)$$

$$P_{j+1}^n = P_{j+1} \circ P_{j+1}^{n-1} = P_{j+1} \circ P_{j+1} \circ \dots \circ P_{j+1} = P_{j+1}, \quad (4-7)$$

$$P_{j+1} = T_j \circ T_{j-1} \circ \dots \circ T_{j-k+1}, \quad (4-8)$$

$$x_{j+1} = P_{j+1} x_{j+1-k} = P_{j+1} x_{j+1}, \quad (4-9)$$

Definition 4.2 An iterative periodic mapping with period k is called “the periodic g -contractive mapping with period k ”, if $\exists G(\text{constant}) < 1$, such that

$$s_n \leq G < 1, \forall n. \quad (4-10)$$

Theorem 4-3 (Theorem 2.8 of [10]). Any periodic g -contractive mapping with period k on complete nonempty metric space X has a unique set of k related fixed points x_j^* in X .

$$x_j^* = P_j x_j^*, T_j x_j^* = x_{j+1}^*, x_j^* = x_{j+k}^*, j = 1, 2, \dots, k. \quad (4-11)$$

Definition 4.4 For a periodic g-contractive mapping with period k, $T_i = T_{j+nk}$, then,

$$T_j \equiv \lim_{i \rightarrow \infty} T_i, \tag{4-12}$$

Proof: Let

$$i = j + nk, j = 1, 2, \dots, k. \tag{4-13}$$

One-to-one corresponding holds for both sides of (4-13)

$$\lim_{i \rightarrow \infty} T_i = \lim_{n \rightarrow \infty} T_{j+nk} = T_j, \tag{4-14}$$

$$x_i = x_{j+nk} = P_j^n x_j,$$

Choose an initial point, say $x_j \in X_j$, then

$$x_j \equiv \lim_{i \rightarrow \infty} x_i = \lim_{n \rightarrow \infty} P_j^n x_j = P_j x_j, \tag{4-14}$$

From any initial point x_j , one can find all x_j^* by (4-11) if (4-10) holds. However, it is still hard to make sure the existence or find out the constant G such that (4-10) holds, especially for $n \rightarrow \infty$.

Definition 4.5 The limit of g-contractive ratio s_i at $i \rightarrow \infty$ is defined by (4-15).

$$\lim_{i \rightarrow \infty} s_i^j = s_j^j, j = 1, 2, \dots, k. \tag{4-15}$$

Proof: By (4-2), (4-3), (4-5), we have

$$s_j^j = (r_j r_{j-1} \dots r_1) = \frac{\|T_j x_j - T_{j-1} x_{j-1}\|}{\|T_1 x_1 - x_1\|} = \frac{\|T_{j+nk} x_{j+nk} - T_{j-1+nk} x_{j-1+nk}\|}{\|T_1 x_1 - x_1\|} = s_{j+nk}^{j+nk}, \tag{4-16}$$

Taking limit on both sides of (4-16), and let $i = j + nk$ we have

$$\lim_{i \rightarrow \infty} s_i^j = \lim_{n \rightarrow \infty} s_{j+nk}^{j+nk} = \lim_{n \rightarrow \infty} s_j^j = s_j^j, \tag{4-17}$$

There are limited terms of s_j^j , let

$$G = \max_j s_j^j, j = 2, 3, \dots, k. \tag{4-18}$$

If $G < 1$, then one can find k related fixed points on X; Otherwise, the method fails.

5 Conclusion

Periodic phenomenon is an universal phenomenon which existed in nature and bio-circle (including human society) long-long ago. However, the updated description on periodic phenomenon by the definition of limit of periodic function at infinity conflicts with the nature at two points: one is the limit does not exist; the other is the limit can only be one if it exists. In this paper, the value of an initial point of periodic function (periodic iterative g-contractive mapping as well) is used as the definition of limit of periodic function at infinity based on a transformation of variables. Furthermore, the limit of g-contractive ratio at infinity is defined by similar transformation of variables. By which, one can find k-related fixed points based on

periodic iterative g -contractive mapping theorem. Application of these definitions, one can describe the cycling path problem, i.e., the “egg comes first or chicken comes first” problem, the k -polar (extreme points) problems (in bio-cycle, $k=1$, viviparous animals; $k=2$, egg and bird, $k=3$, egg, polliwog, and frog), and to calculate the limit of combinations of periodic functions at infinity.

Competing Interests

Author has declared that no competing interests exist.

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