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Distribution Solutions for Impulsive Evolution Partial Differential Equations

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Abstract

In this paper, the Solution to Impulsive Evolution Partial Differential Equations is given by applying the Distribution Technigue and Rankine-Hugoniot condition.

Keywords: Distribution; impulsive; evolutionary process and rankine-hugoniot condition.

2010 Mathematics Subject Classification: 35R12, 49N25, 60E05

1 Introduction

Various evolutionary processes in real life situations or physical phenomena encountered in the diverse fields of human endeavour, like biotechnology, industrial robotics, pharmcokinetics, optimal control and population dynamics are characterised by the fact that they usually undergo abrupt changes of state at certain moment of time between intervals of continuous evolution. Such changes due to its time-lag, which are often negligible compared to the total duration of the process, are regarded as having acted instantaneously and in the form of impulses.

As a consequence, impulsive differential equations have been developed for modeling of impulsive problems. In recent times, much attention had been given by researchers impulsive problems in differet directions. These directions indicate problems in biological and social macro systems which have much functional applications as a result of processes that involves hereditary issues and this had been worked upon by many mathematicians (see for example [1], [2]). Sometimes, the derivatives of

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the state variable may undergo a delay, such is being handled by introduction of neutral functional differential equations and inclusions (see for example [3,4]).

Others have initiated the study of the state dependent delay (see for example [5,6]) while sudies had been done also on impulsive evolutionary equations (see for example [7,8]) and in particular,[9] considered the solution to impulsive evolution equations by the method of differential inequalities via the Lyapunov functions.

Nevertheless, uptill now to the best our knowledge, the solution obtained by distribution techniques is an untreated problem and this is the motivation for this paper. We consider the nonlinear impulsive evolution equation of the form:

$$u_t + (\Psi(u))_x = 0 \quad \text{on} \quad \mathbb{R}\backslash C([t_k, t_{k+1}]) \tag{1.1}$$

$$\Delta u(t_k, x) = I_k(u(t_k, x)), \quad k = 1, 2, ..., n \tag{1.2}$$

$$u(0, x) = u_0(x) \tag{1.3}$$

where $I_k \in C(\mathbb{R}, \mathbb{R})$, $\Psi : \mathbb{R} \to \mathbb{R}^n$ is a smooth function,

$$\Delta u|_{t=t_k} = u(t_k^+, x) - u(t_k^-, x); \quad u(t_k^+, x), u(t_k^-, x)$$

indicate the right and left limits of the function u at t_k. By using the idea of Rankine-Hugoniot condition and the distribution approach, we give the solution representation to problem 1.1 - 1.3.

2 Preliminaries

Definition 2.1. : A function $u : J \to \mathbb{R}$ is said to be piecewise continuous denoted by $PC(J, \mathbb{R})$ if u is continuous at each point of the in interval $J = (\alpha, \beta)$ except for some t_k for which $u(t_k^+)$ and $u(t_k^-)$ exist.

Moreso, impulses problem requires a generalised definition of functions given that we often arrived at unit intervals being supported on shorter time-lags or rather the functions could be regarded as an instantteneous unit impulse. Such generalised definition are often necessary for problems involving idealised point singularities in that it weakens the strict requirement of any solution being continuously differentiable upto the order of the differential equation.[3mm] Let $\Omega \subseteq [t_k, t_{k+1}] \times \mathbb{R}^n$ be nonempty open set and ϕ a smooth test function belonging to $D(\Omega) = C_0^\infty(\Omega)$. We define thus:

Definition 2.2. ([10], p 131): A distribution or generalised function is a linear mapping $\phi \to (f, \phi)$ from $D(\Omega)$ to $\mathbb{R}$ which is continuous in the sense that if $\phi_n \to \phi$ in $D(\Omega)$, then $(f, \phi_n) \to (f, \phi)$ where

$$(f, \phi) := \int_\Omega f(x)\phi(x)dx \quad \forall x \in \Omega$$

.

Definition 2.3. (Rankine-Hugoniot condition): Let N be an open neigbourhood in the open upper half plane and suppose a curve $C : (\alpha, \beta) \ni t \to C(t)$ divides N into two pieces, N^L and N^r, lying to the left and right of the curve respectively. Suppose u undergoes discontinuities denoted by [u] at the curve C which are continuous along C, then for any point $p \in C$, the gradient S of C at p have the relation $S[u] = [\Psi(u)]$ where $[u](p) = u^r(p) - u^l(p) = \lim_{(t^r,x^r)\to p} u(t^r, x^r) - \lim_{(t^l,x^l)\to p} u(t^l, x^l)$ and Ψ as in 1.1, indicates the rate (velocities) at any point p.

Consider the problem 1.1 - 1.3, assume $\Omega_\pm$ to be the space of discontinuities in the interval $[t_k, t_{k+1}], \quad k = 1, 2, ..., n$ defined by

$$\Omega_\pm = \begin{cases} & x > C(t_k) \\ (t,x) \in \Omega : & \\ & x < C(t_k). \end{cases} \tag{2.1}$$

The outward normal along $C(t_k)$ is given as

$$n(t_k) = \frac{\pm(C'(t_k), -1)}{\sqrt{1 + C'(t_k)^2}} \tag{2.2}$$

where the surface measure along C is

$$d\alpha(t_k) = \sqrt{1 + C'(t_k)^2}dt \tag{2.3}$$

We then apply these definitions and the divergence theorem to obtain the solution representation by multiplying the problem with the test function and finding the inner product.

3 Main Results

Let $\phi \in C_0^2(\Omega)$ be a test function, we multiply equation 1.1 with ϕ and integrate accordingly by part to obtain

$$\int\limits_{\Omega} [(u_t, \phi) + (\Psi(u)u_x, \phi)]\, dxdt = 0 \tag{3.1}$$

$$\Rightarrow \int\limits_{\mathbb{R}} \int\limits_{t \geq 0} (u\phi_t + \Psi(u)\phi_x)dtdx - \int\limits_{\mathbb{R}} |u\phi| \Big|_{t=0}^{\infty} dx = 0$$

.

Definition 3.1. A bounded measurable function $u(t,x) \in C'(\Omega \backslash C([t_k, t_{k+1}]))$ is called a distribution solution of 1.1 - 1.3 if it has the form

$$\int\limits_{\mathbb{R}} \int\limits_{t \geq 0} [u(t,x)\phi_t(t,x) + \Psi(u(t,x))\phi_x(t,x)]dtdx + \int\limits_{\mathbb{R}} u_0(x)\phi(0,x))dx = 0 \tag{3.2}$$

such that

$$C'(t)[u^+(t, C(t)) - u^-(t, C(t))] = \Psi(u^+(t, C(t)) - \Psi((u^-(t, C(t)) \tag{3.3}$$

$\forall t \in [t_k, t_{k+1}]$.

Equation 3.2 follows from integration by part and applying the divergince theorem, we have that

$$\begin{aligned} \int\limits_{\Omega} (u\phi_t + \Psi(u)\phi_x)dtdx &= \int\limits_{\Omega_\pm} (u, \Psi(u).(\phi_t, \phi_x)dxdt = 0 \\ &= \int\limits_{\partial\Omega_\pm} \phi(u, \Psi(u).n(t)d\alpha(t) \end{aligned} \tag{3.4}$$

which implies

$$\pm \int\limits_{t_k}^{t_{k+1}} [u_t^\pm(t, C(t))C'(t) - \Psi(u(t, C(t))]\phi(t, C(t))dt \tag{3.5}$$

follwing the Rankine-Hugoniot condition.

Substituting equation 3.5 into the problem gives

$$\int_{t_k}^{t_{k+1}} \{C'(t)[u^+(t,C(t)) \quad - \quad u^-(t,C(t))] - \Psi(u^+(t,C(t)) \\ -\Psi((u^-(t,C(t))\}\phi(t,C(t)dt = 0 \qquad (3.6)$$

for all ϕ and then

$$C'(t)[u^+(t,C(t)) - u^-(t,C(t))] - (\Psi(u^+(t,C(t)) - \Psi((u^-(t,C(t))) = 0$$

thus

$$C'(t)[u^+(t,C(t)) - u^-(t,C(t))] = \Psi(u^+(t,C(t)) - \Psi((u^-(t,C(t)).$$

The characteristics curve along or within the impulse range can then be obtained from the equation

$$C'(t) = \frac{\Psi(u^+(t,C(t)) - \Psi((u^-(t,C(t))}{u^+(t,C(t)) - u^-(t,C(t))} \qquad (3.7)$$

<u>Illustration</u>: Consider the impulsive Burger's equation of the form:

$$\begin{aligned} u_t + uu_x &= 0 \quad \text{on} \quad -\infty < x < \infty, \quad 0 \le t < \infty \\ \Delta u(t_k, x) &= I_k(u(t_k, x)) \\ u(0,x) &= u_0(x) = \begin{cases} 0 & x \ge 1 \\ 1-x & 0 < x < 1 \\ 1 & x \le 0. \end{cases} \end{aligned}$$

The characteristics equations arising from the above conditions starting from initial point $(0, x_0)$ are

$$\begin{aligned} x(t) &= (1-x_0)t + x_0 && \text{if} \quad x_0 \in (0,1) \\ x(t) &= x_0 + t && \text{if} \quad x_0 < 0 \\ x(t) &= x_0 && \text{if} \quad x_0 > 1. \end{aligned}$$

along which the concentration $u(t, x)$ remain constant.

Our concern is to determine a distributional solution valid for all (t, x) in the region of discontinuity where the impulse effect occurs. Thus, we consider a characteristic curve $C(t)$ in Ω such that for $x > C(t), \quad u = 0$ and $x < C(t), \quad u = 1$, in the region $\mathbb{R} = \{(t,x) : t \le 1, \quad x \ge 1 \quad \text{and} \quad x \le t\}$.

The continuity conditions implies that

$$\begin{aligned} C'(t)(0-1) &= (0 - \tfrac{1}{2}) \quad = -\tfrac{1}{2} \\ \Rightarrow \quad C'(t) &= \tfrac{1}{2} \end{aligned}$$

hence the characteristics equation becomes

$$C(t) = \frac{1}{2}t + 1, \quad t \ge 0$$

for which the solution is distributed above and below.

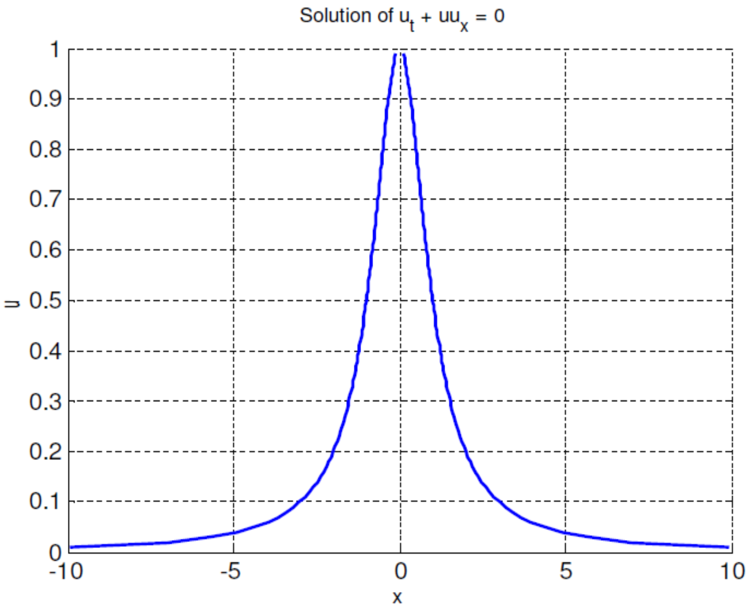

Fig. 1. A Plot of u vs x

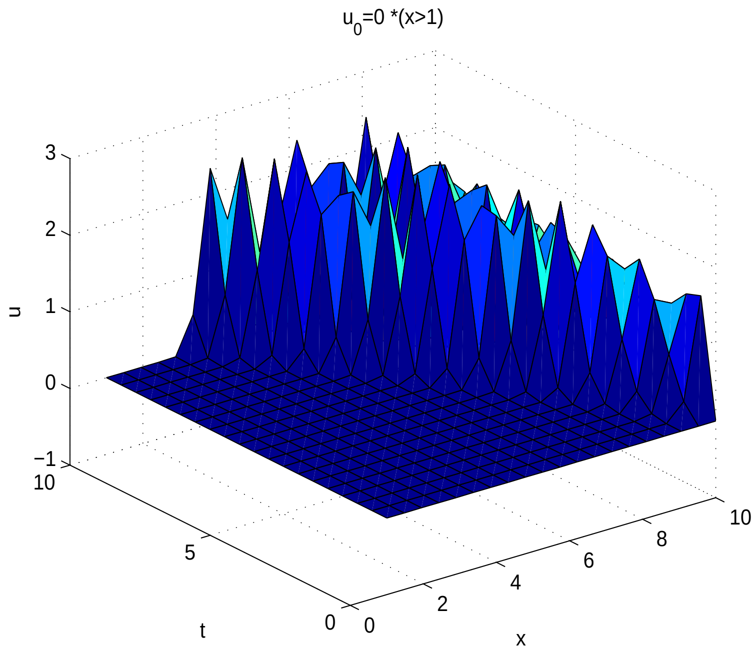

Fig. 2. Surface Plot (x, t, u)

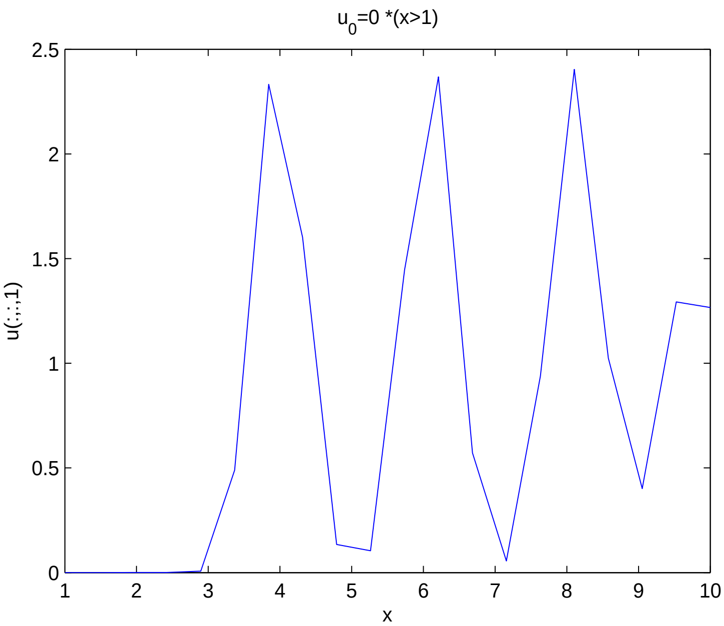

Fig. 3. A Plot of x vs $u(:,:,1)$

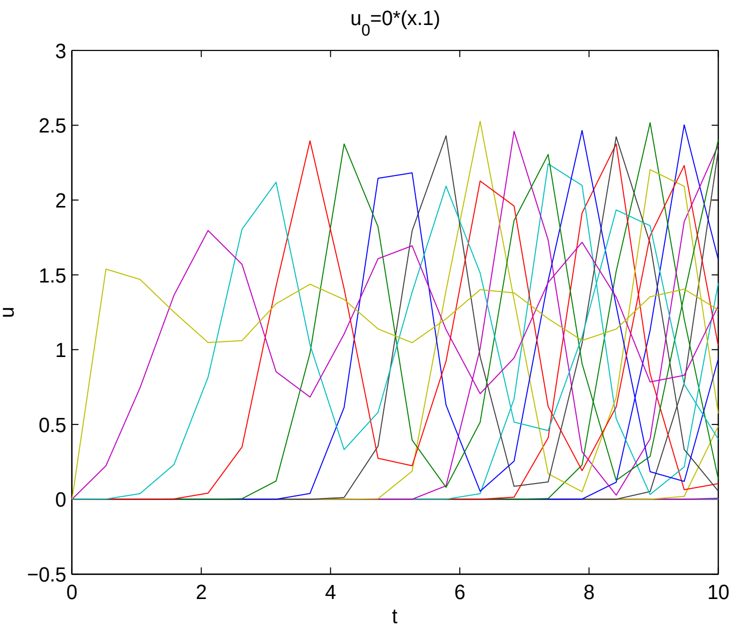

Fig. 4. A Plot of x vs u

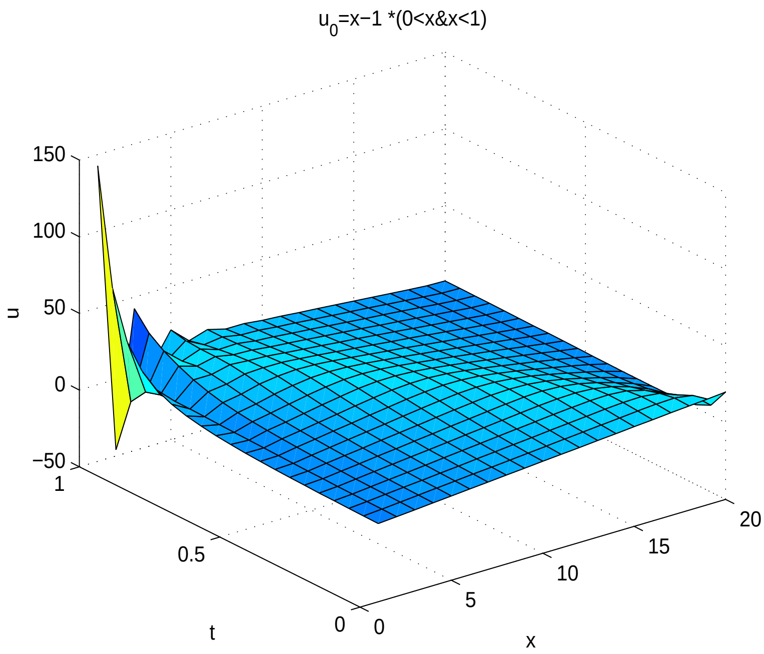

Fig. 5. Surface Plot (x, t, u)

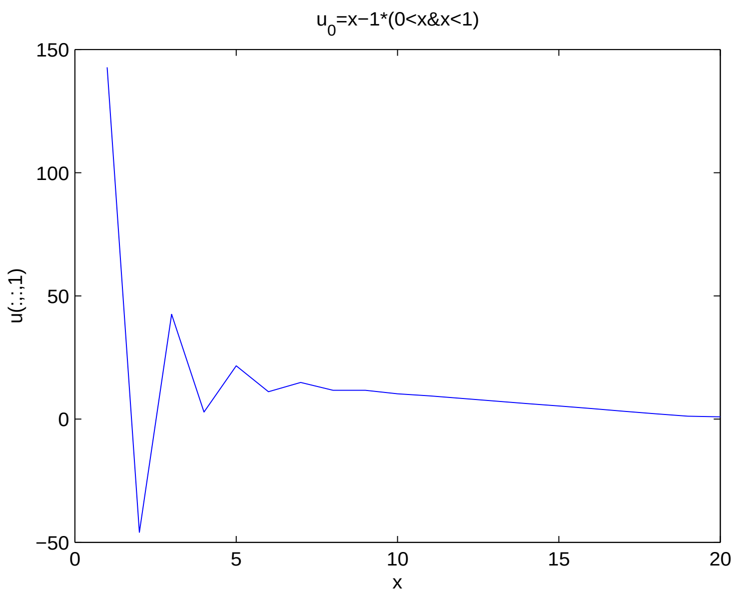

Fig. 6. A Plot of x vs $u(:, :, 1)$

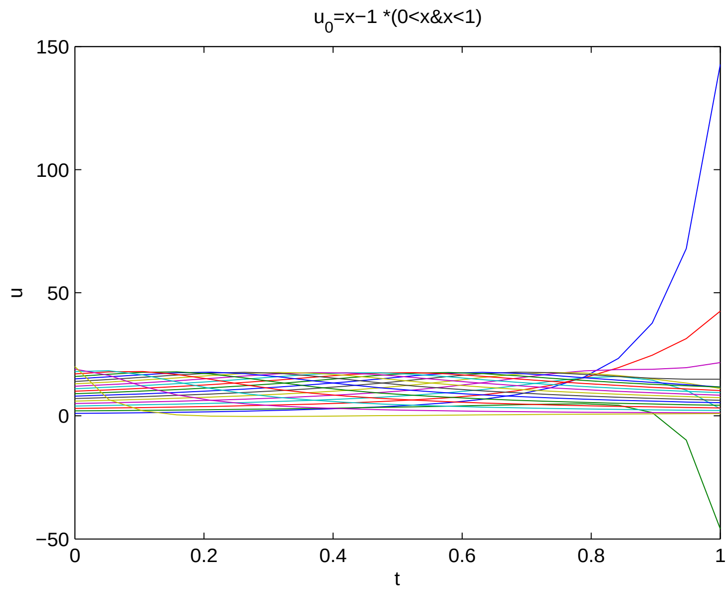

Fig. 7. A Plot of x vs u

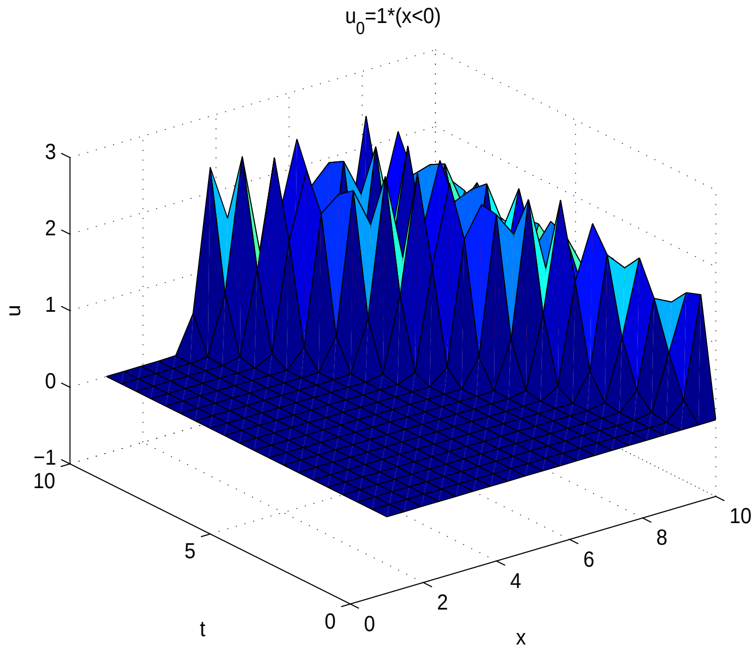

Fig. 8. Surface Plot (x, t, u)

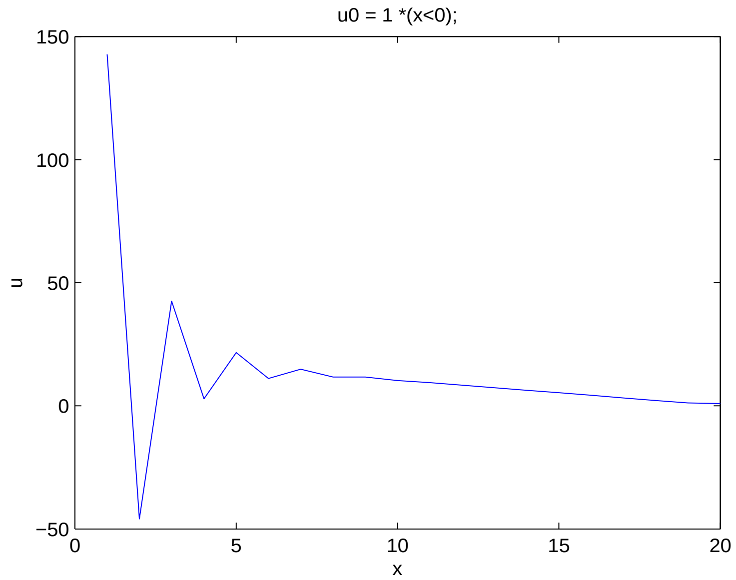

Fig. 9. A Plot of x vs $u(:,:,1)$

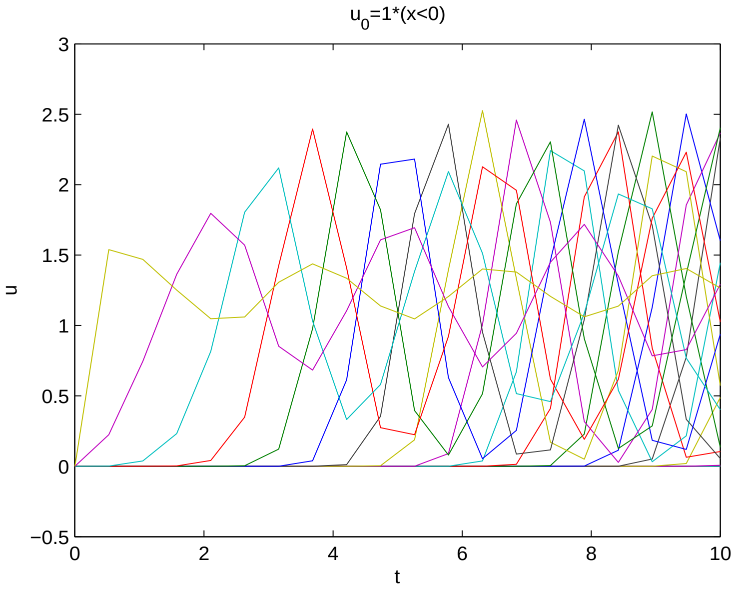

Fig. 10. A Plot of x vs u

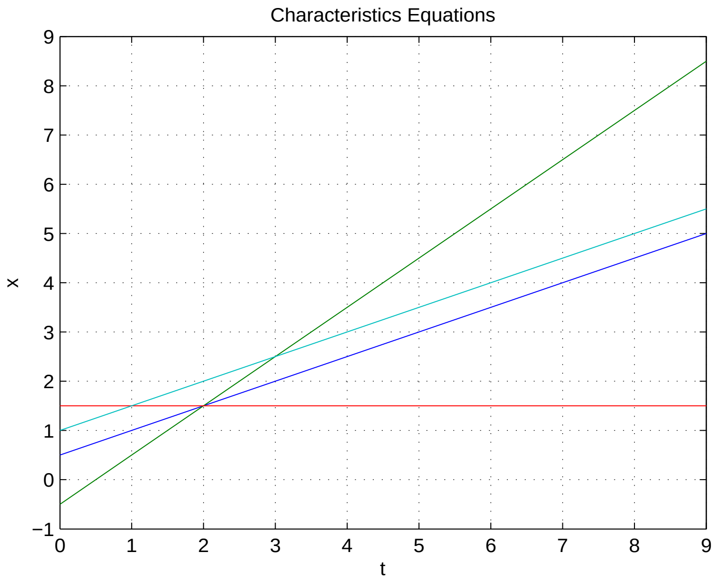

Fig. 11. A Plot of t vs x

4 Conclusion

The Figs. (1) - (11) clearly show the region of impulses at the point of discontinuities t_k.

Competing Interests

The authors declare that no competing interests exist.

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