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Almost Periodic Sequence in a Discrete Logistic Equation

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

This paper is concerned with an almost periodic discrete logistic equation. By using the continuation theorem of Mawhin's coincidence degree theory, this paper investigates the existence and stability of a unique positive almost periodic sequence solution of the equation. These results generalize and improve the previous works, and they are easy to check. An example with a numerical simulation is also given to demonstrate the effectiveness of the results in this paper.

Keywords: Almost periodicity; coincidence degree; logistic; discrete.

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1 Introduction

Let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of real numbers, integers and positive integers, respectively. Related to a continuous bounded function f on $\mathbb{E}\,(\mathbb{E} = \mathbb{R}$ or $\mathbb{Z})$, we use the following notations:

$$f^- = \inf_{s\in\mathbb{E}} f(s), \quad f^+ = \sup_{s\in\mathbb{E}} f(s), \quad |f|_\infty = \sup_{s\in\mathbb{E}} |f(s)|.$$

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Discrete time models governed by difference equation are more appropriate than the continuous ones when the populations have non-overlapping generations. Moreover, since the discrete time models can also provide efficient computational models of continuous models for numerical simulations, it is reasonable to study discrete time models governed by difference equations. The purpose of this article is to investigate the dynamics of a discrete logistic equation:

$$x(n+1) = x(n)\exp\left(r(n)\left[1-\frac{x(n)}{K(n)}\right]\right), \tag{1.1}$$

where $\{r(n)\}$ and $\{K(n)\}$ are bounded nonnegative almost periodic sequences with $r^- > 0$ and $K^- > 0$.

In [1], Mohamad and Gopalsamy proposed Eq. (1.1) and studied the existence and stability of a positive almost periodic solution of the model. They obtained

Theorem 1.1. ([1]) *If $r^+ < 2$, then Eq. (1.1) has a unique globally asymptotically stable almost periodic solution.*

Following, by using asymptotically almost periodic theory, Li and Chen [2] also investigated the existence and stability of a positive almost periodic solution of Eq. (1.1). They obtained

Theorem 1.2. ([2]) *If $\frac{K^+}{K^-}\exp(r^+-1) < 2$, then Eq. (1.1) has a unique globally asymptotically stable almost periodic solution.*

By applying the continuation theorem of Mawhin's coincidence degree theory and some analysis techniques, the purpose of this article is to investigate the dynamics of Eq. (1.1) and one gets that

Theorem 1.3. *Eq. (1.1) has a unique globally asymptotically stable almost periodic solution.*

Remark 1.1. Without $\frac{K^+}{K^-}\exp(r^+-1) < 2$ and $r^+ < 2$, Theorem 1.3 is simpler than Theorems 1.1-1.2. Therefore, the results in this paper generalize and improve the previous works in [1-2].

The organization of this paper is as follows. In Section 2, we give some definitions and present some useful lemmas. In Section 3, by using Mawhin's continuation theorem of coincidence degree theory and constructing a suitable Lyapunov functional, we establish sufficient conditions for the existence of a unique globally asymptotically stable almost periodic solution of Eq. (1.1). An example with a numerical simulation is also given to demonstrate the effectiveness of the results in this paper.

2 Preliminaries

Definition 2.1. ([3]) Let $f \in C(\mathbb{R})$. f is said to be almost periodic function on $\mathbb{R}$, if for $\forall\epsilon > 0$, the set

$$T(f,\epsilon) = \{\tau : |f(t+\tau) - f(t)| < \epsilon, \forall t \in \mathbb{R}\}$$

is relatively dense, i.e., for $\forall\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval length l, there exists a number $\tau = \tau(\epsilon) \in T(f,\epsilon)$ in this interval such that

$$|f(t+\tau) - f(t)| < \epsilon, \quad \forall t \in \mathbb{R}.$$

τ is called to the ϵ-almost period of f, $T(f,\epsilon)$ denotes the set of ϵ-almost periods for f and $l(\epsilon)$ is called to the length of the inclusion interval for $T(f,\epsilon)$. Let $AP(\mathbb{R})$ denote the set of all real valued almost periodic functions on $\mathbb{R}$.

Definition 2.2. ([4, 5, 6]) A sequence $f : \mathbb{Z} \to \mathbb{R}$ is called an almost periodic sequence if the ϵ-translation set of f

$$E\{f, \epsilon\} = \{\tau \in \mathbb{Z} : |f(n+\tau) - f(n)| < \epsilon, \quad \forall n \in \mathbb{Z}\}$$

is a relatively dense set in $\mathbb{Z}$ for all $\epsilon > 0$; that is, for any given $\epsilon > 0$, there exists an integer $l(\epsilon) > 0$ such that each interval of length $l(\epsilon)$ contains an integer $\tau \in E\{f, \epsilon\}$ such that

$$|f(n+\tau) - f(n)| < \epsilon, \quad \forall n \in \mathbb{Z}.$$

τ is called the ϵ-translation number or ϵ-almost period. Let $AP(\mathbb{Z})$ denote the set of all real valued almost periodic sequences.

We narrate a number of results on almost periodic sequences for the benefit of the reader. The proofs of the following results can be found in Samoilenko and Perestyuk [7].

Lemma 2.1. *An almost periodic sequence is bounded.*

Lemma 2.2. *If $f \in AP(\mathbb{Z})$ and $g \in AP(\mathbb{Z})$ with $g^- > 0$, then $f + g, fg, \frac{f}{g} \in AP(\mathbb{Z})$.*

Lemma 2.3. *If $f \in AP(\mathbb{Z})$, $g \in AP(\mathbb{Z})$ and $\epsilon > 0$ is an arbitrary real number, then there exists a relatively dense set of their common ϵ-almost periods.*

Next, we present and prove several useful lemmas which will be used in later section.

Let $f \in AP(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\mathbb{Z})$, for $\forall\epsilon > 0$, set

$$T_\epsilon^+ = \{t \in \mathbb{R}\backslash\mathbb{Z} : f(t) \in [f^* - \epsilon, f^*], f'(t) \geq 0\},$$
$$T_\epsilon^- = \{t \in \mathbb{R}\backslash\mathbb{Z} : f(t) \in [f_*, f_* + \epsilon], f'(t) \leq 0\},$$

where $f^* = \sup_{s\in\mathbb{R}} f(s)$, $f_* = \inf_{s\in\mathbb{R}} f(s)$.

Lemma 2.4. *For $\forall\epsilon > 0$, $T_\epsilon^+ \neq \emptyset$.*

Proof. By the way of contradiction. Suppose that $T_\epsilon^+ = \emptyset$ for some $\epsilon > 0$. Then we have

$$f'(t) < 0, \quad \forall t \in \{t \in \mathbb{R}\backslash\mathbb{Z} : f(t) \in [f^* - \epsilon, f^*]\}. \tag{2.1}$$

Since $f \in AP(\mathbb{R})$, there must exist a point t_0 such that $f(t_0) = f^* - \epsilon$. It follows from (2.1) that

$$f(t) \leq f^* - \epsilon, \quad \forall t \geq t_0. \tag{2.2}$$

By the definition of f^* and (2.2), there is a point $t_1 \in (-\infty, t_0)$ such that

$$f(t_1) \in \left[f^* - \frac{\epsilon}{2}, f^*\right]. \tag{2.3}$$

For $\frac{\epsilon}{4} > 0$, it is possible to find a real number $l = l(\frac{\epsilon}{4}) > 0$, there exists a number $\tau = \tau(\frac{\epsilon}{4}) \in [t_0 - t_1, t_0 - t_1 + l]$ such that $|f(t+\tau) - f(t)| < \frac{\epsilon}{4}$, $\forall t \in \mathbb{R}$, which implies that

$$|f(t_1 + \tau) - f(t_1)| < \frac{\epsilon}{4},$$

which yields from (2.3) that

$$f(t_1 + \tau) > f(t_1) - \frac{\epsilon}{4} \geq f^* - \frac{3\epsilon}{4}. \tag{2.4}$$

Since $t_1 + \tau \geq t_0$, we have from (2.2) that $f(t_1 + \tau) \leq f^* - \epsilon$, which leads to a contradiction with (2.4). Therefore, for $\forall\epsilon > 0$, $T_\epsilon^+ \neq \emptyset$. This completes the proof. $\square$

Similar to Lemma 2.4, we can easily show that

Lemma 2.5. *For* $\forall\epsilon > 0$, $T_\epsilon^- \neq \emptyset$.

Lemma 2.6. *Assume that* $f \in AP(\mathbb{Z})$ *is a solution of the following equation:*

$$f(n+1) = f(n)\exp\left(r(n)\left[1 - \frac{f(n)}{K(n)}\right]\right).$$

Define

$$\vec{f}(t) = f(n)\exp\left(r(n)\left[1 - \frac{f(n)}{K(n)}\right](t-n)\right), \quad \forall t \in [n, n+1),\ n \in \mathbb{Z}. \tag{2.5}$$

Then $\vec{f} \in AP(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\mathbb{Z})$ *is a solution of the following system:*

$$\frac{\mathrm{d}\,\vec{f}(t)}{\mathrm{d}\,t} = \vec{f}(t)r(n)\left[1 - \frac{\vec{f}(n)}{K(n)}\right], \quad \forall t \in (n, n+1),\ n \in \mathbb{Z}. \tag{2.6}$$

Furthermore, one has $\vec{f}_* \leq f_* \leq f^* \leq \vec{f}^*$.

Proof. Obviously, $\vec{f}(n) = f(n)$, $\forall n \in \mathbb{Z}$. So $\vec{f}_* \leq f_* \leq f^* \leq \vec{f}^*$.

Next, we should prove the remaining part of this lemma by three steps.

Firstly, we claim that $\vec{f} \in C(\mathbb{R})$. Clearly, $\vec{f} \in C(\mathbb{R}\backslash\mathbb{Z})$. For $t = n \in \mathbb{Z}$, in view of (2.5), one has $\vec{f}(t) = \vec{f}(n) = f(n)$ and

$$\begin{aligned}\lim_{t\to n^-} \vec{f}(t) &= \lim_{t\to n^-} f(n-1)\exp\left(r(n-1)\left[1 - \frac{f(n-1)}{K(n-1)}\right](t-n+1)\right)\\ &= \lim_{t\to n^-} f(n-1)\exp\left(r(n-1)\left[1 - \frac{f(n-1)}{K(n-1)}\right]\right) = f(n).\end{aligned}$$

So $\vec{f} \in C(\mathbb{R})$.

Secondly, we claim that $\vec{f} \in AP(\mathbb{R})$. For convenience, let

$$F(n) = r(n)\left[1 - \frac{f(n)}{K(n)}\right], \quad \forall n \in \mathbb{Z}.$$

By the almost periodicity of r, f and K, $F \in AP(\mathbb{Z})$. Let $G = \max\{e^{F^+}, f^+e^{F^+}\}$. Since $f, F \in AP(\mathbb{Z})$, for $\forall\epsilon > 0$, there exists an integer $l(\frac{\epsilon}{2G}) > 0$ such that each interval of length $l(\frac{\epsilon}{2G})$ contains an integer $\tau \in E\{f, \frac{\epsilon}{2G}\} \cap E\{F, \frac{\epsilon}{2G}\}$ such that

$$|\vec{f}(n+\tau) - \vec{f}(n)| = |f(n+\tau) - f(n)| < \frac{\epsilon}{2G}, \quad |F(n+\tau) - F(n)| < \frac{\epsilon}{2G}, \quad \forall n \in \mathbb{Z}. \tag{2.7}$$

For arbitrary $t \in (n, n+1)$, we have $t + \tau \in (n+\tau, n+\tau+1)$, $n \in \mathbb{Z}$. Then

$$\begin{aligned}|\vec{f}(t+\tau) - \vec{f}(t)| &= \Big|f(n+\tau)\exp\big[F(n+\tau)(t-n)\big] - f(n)\exp\big[F(n)(t-n)\big]\Big|\\ &\leq |f(n+\tau) - f(n)|\exp\big[F(n+\tau)(t-n)\big]\\ &\quad +|f(n)|\Big|\exp\big[F(n+\tau)(t-n)\big] - \exp\big[F(n)(t-n)\big]\Big|\\ &\leq e^{F^+}|f(n+\tau) - f(n)| + f^+e^{\xi}\Big|F(n+\tau)(t-n) - F(n)(t-n)\Big|\\ &< \frac{e^{F^+}}{2G}\epsilon + \frac{f^+e^{F^+}}{2G}\epsilon \leq \epsilon.\end{aligned} \tag{2.8}$$

From (2.7)-(2.8), $\tau \in T(\vec{f}, \epsilon)$. Therefore, $\vec{f} \in AP(\mathbb{R})$.

Finally, for $t \in (n, n+1)$, $n \in \mathbb{Z}$, it follows from (2.6) that

$$\frac{\mathrm{d}\,\vec{f}(t)}{\mathrm{d}\,t} = f(n)\exp\left(r(n)\left[1 - \frac{f(n)}{K(n)}\right](t-n)\right)r(n)\left[1 - \frac{f(n)}{K(n)}\right] = \vec{f}(t)r(n)\left[1 - \frac{\vec{f}(n)}{K(n)}\right].$$

In sum, $\vec{f} \in AP(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\mathbb{Z})$ is a solution of (2.6). This completes the proof. $\square$

Similar to the proof of the above lemma, one could easily show that

Lemma 2.7. *Assume that $f \in AP(\mathbb{Z})$, define*

$$\tilde{f}(t) = [f(n+1) - f(n)](t-n) + f(n), \quad \forall t \in [n, n+1),\ n \in \mathbb{Z}. \tag{2.9}$$

Then $\tilde{f} \in AP(\mathbb{R})$ and $\|f\|_{\mathbb{Z}} = \|\tilde{f}\|_{\mathbb{R}}$.

3 Main Results

The method to be used in this paper involves the applications of the continuation theorem of coincidence degree. This requires us to introduce a few concepts and results from Gaines and Mawhin [8].

Let $\mathbb{X}$ and $\mathbb{Y}$ be real Banach spaces, $L : \mathrm{Dom}L \subseteq \mathbb{X} \to \mathbb{Y}$ be a linear mapping and $N : \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\mathrm{Im}L$ is closed in $\mathbb{Y}$ and $\mathrm{dimKer}L = \mathrm{codimIm}L < +\infty$. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I - Q)$. It follows that $L|_{\mathrm{Dom}L\cap\mathrm{Ker}P} : (I - P)\mathbb{X} \to \mathrm{Im}L$ is invertible and its inverse is denoted by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to \mathbb{X}$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

Lemma 3.1. ([8]) *Let $\Omega \subseteq \mathbb{X}$ be an open bounded set, L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. If all the following conditions hold*:

(a) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap \mathrm{Dom}L$, $\lambda \in (0, 1)$;

(b) $QNx \neq 0$, $\forall x \in \partial\Omega \cap \mathrm{Ker}L$;

(c) $\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$, *where $J : \mathrm{Im}Q \to \mathrm{Ker}L$ is an isomorphism.*

Then $Lx = Nx$ has a solution on $\bar{\Omega} \cap \mathrm{Dom}L$.

Now we are in the position to present and prove our result on the existence of at least one positive almost periodic sequence solution of Eq. (1.1).

For $f \in AP(\mathbb{Z})$, define $\|f\|_{\mathbb{Z}} = \sup_{s\in\mathbb{Z}} |f(s)|$ and denote by

$$\bar{f} = m(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(k)$$

the mean value of f. For $f \in AP(\mathbb{R})$, define $\|f\|_{\mathbb{R}} = \sup_{s\in\mathbb{R}} |f(s)|$ and denote by

$$\Lambda(f) = \left\{\theta \in \mathbb{R} : \lim_{T\to\infty} \frac{1}{T} \int_0^T f(s)e^{-\mathrm{i}\theta s}\mathrm{d}s \neq 0\right\}$$

the set of Fourier exponents of f.

Theorem 3.2. *Eq.* (1.1) *admits at least one positive almost periodic sequence solution.*

Proof. Under the invariant transformation $x = e^y$, Eq. (1.1) reduces to

$$y(n+1) = y(n) + r(n)\left[1 - \frac{e^{y(n)}}{K(n)}\right]. \tag{3.1}$$

It is easy to see that if Eq. (3.1) has one almost periodic sequence solution y, then $x = e^y$ is a positive almost periodic sequence solution of Eq. (1.2). Therefore, to completes the proof it suffices to show that Eq. (3.1) has one almost periodic sequence solution.

Take $\mathbb{X} = \mathbb{Y} = \mathbb{V} \bigoplus \mathbb{R}$, where

$$\mathbb{V} = \left\{ y \in AP(\mathbb{Z}) : \forall\theta \in \Lambda(\tilde{y}), \theta \in \Theta \right\},$$

where $\tilde{y}$ is defined as that (2.9), $\Theta = \bigcup_{k\in\mathbb{K}}[2k\pi + \rho, 2(k+1)\pi - \rho]$, $\mathbb{K} = \{k_1, k_2, \dots, k_n\} \subset \mathbb{Z}$, $n \in \mathbb{N}$ and $\rho \in (0, \frac{\pi}{2})$ are given constant. Define the norm

$$\|y\|_{\mathbb{Z}} = \sup_{s\in\mathbb{Z}} |y(s)|, \quad \forall y \in \mathbb{X} = \mathbb{Y},$$

then $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces with the norm $\|\cdot\|_{\mathbb{Z}}$ (see Lemma 5.2 in section 5). Set

$$L : \mathbb{X} \to \mathbb{Y}, \quad Ly(n) = \Delta y(n) = y(n+1) - y(n), \quad \forall y \in \mathbb{X}$$

and

$$N : \mathbb{X} \to \mathbb{Y}, \quad Ny = \left[r(n)\left[1 - \frac{e^{y(n)}}{K(n)}\right]\right].$$

With these notations Eq. (3.1) can be written in the form

$$Ly = Ny, \quad \forall y \in \mathbb{X}.$$

It is not difficult to verify that $\mathrm{Ker}L = \mathbb{R}$, $\mathrm{Im}L = \mathbb{V}$ is closed in $\mathbb{Y}$ and $\dim \mathrm{Ker}L = 1 = \mathrm{codim}\,\mathrm{Im}L$. Therefore, L is a Fredholm mapping of index zero (see Lemma 5.3 in section 5). Now define two projectors $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ as

$$Py = m(y) = Qy, \quad \forall y \in \mathbb{X} = \mathbb{Y}.$$

Then P and Q are continuous projectors such that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I - Q)$. Furthermore, through an easy computation we find that the inverse $K_P : \mathrm{Im}L \to \mathrm{Ker}P \cap \mathrm{Dom}L$ of L_P has the form

$$K_P y = \sum_{k=0}^{n-1} y(k) - m\left[\sum_{k=0}^{n-1} y(k)\right], \quad \forall y \in \mathrm{Im}L.$$

Then $QN : \mathbb{X} \to \mathbb{Y}$ and $K_P(I - Q)N : \mathbb{X} \to \mathbb{X}$ read

$$QNy = m\left[r(n)\left[1 - \frac{e^{y(n)}}{K(n)}\right]\right],$$

$$K_P(I - Q)Ny = f(y) - Qf(y), \quad \forall y \in \mathrm{Im}L,$$

where $f(y)$ is defined by $f(y) = \sum_{k=0}^{n-1}\left[Ny(k) - QNy(k)\right]$. Then N is L-compact on $\bar{\Omega}$ (see Lemma 5.4 in section 5).

In order to apply Lemma 3.1, we need to search for an appropriate open-bounded subset Ω.

Corresponding to the operator equation $Ly = \lambda y$, $\lambda \in (0,1)$, we have

$$y(n+1) - y(n) = \lambda r(n)\left[1 - \frac{e^{y(n)}}{K(n)}\right]. \tag{3.2}$$

Suppose that $y \in \mathbb{X}$ is a solution of Eq. (3.2) for some $\lambda \in (0,1)$. Then $x = e^y \in AP(\mathbb{Z})$ is a positive solution of the following equation:

$$x(n+1) = x(n)\exp\left(\lambda r(n)\left[1 - \frac{x(n)}{K(n)}\right]\right).$$

By Lemma 2.6, there exists $\vec{x} \in AP(\mathbb{R}, (0,+\infty)) \cap C^1(\mathbb{R}\backslash\mathbb{Z})$ satisfying

$$\frac{\mathrm{d}\,\vec{x}(t)}{\mathrm{d}\,t} = \lambda \vec{x}(t) r(n)\left[1 - \frac{\vec{x}(n)}{K(n)}\right], \quad \forall t \in (n, n+1),\ n \in \mathbb{Z} \tag{3.3}$$

and $\vec{x}_* \le x_* \le x^* \le \vec{x}^*$. Let $\vec{x} = e^{\vec{y}}$ in Eq. (3.3), then

$$\frac{\mathrm{d}\,\vec{y}(t)}{\mathrm{d}\,t} = \lambda r(n)\left[1 - \frac{e^{\vec{y}(n)}}{K(n)}\right], \quad \forall t \in (n, n+1),\ n \in \mathbb{Z}. \tag{3.4}$$

By the definitions of y, x and $\vec{x}$, $\vec{y} \in AP(\mathbb{R})$. By Eq. (3.4), $\vec{y} \in C^1(\mathbb{R}\backslash\mathbb{Z})$.

By Lemma 2.4, for $\forall \epsilon > 0$, $T_\epsilon^+ \ne \emptyset$, that is, there is a point $\xi \in \mathbb{R}\backslash\mathbb{Z}$ such that

$$\vec{y}(\xi) \in [\vec{y}^* - \epsilon, \vec{y}^*] \quad \text{and} \quad \left.\frac{\mathrm{d}\,\vec{y}(t)}{\mathrm{d}\,t}\right|_{t=\xi} \ge 0. \tag{3.5}$$

In view of (3.4), there exists $n_0 \in \mathbb{Z}$ such that $\xi \in (n_0, n_0+1)$ and

$$\left.\frac{\mathrm{d}\,\vec{y}(t)}{\mathrm{d}\,t}\right|_{t=\xi} = \lambda r(n_0)\left[1 - \frac{e^{\vec{y}(n_0)}}{K(n_0)}\right],$$

which implies from (3.5) that

$$\vec{y}(n_0) \le \ln K^+. \tag{3.6}$$

Further, it follows from (3.4) that

$$\int_{n_0}^{\xi} \frac{\mathrm{d}\,\vec{y}(t)}{\mathrm{d}\,t}\,\mathrm{d}t = \int_{n_0}^{\xi} \lambda r(n_0)\left[1 - \frac{e^{\vec{y}(n_0)}}{K(n_0)}\right]\mathrm{d}t \le r^+. \tag{3.7}$$

By (3.6)-(3.7), we have

$$\vec{y}(\xi) = \vec{y}(n_0) + \int_{n_0}^{\xi} \frac{\mathrm{d}\,\vec{y}(t)}{\mathrm{d}\,t}\,\mathrm{d}t \le \ln K^+ + r^+,$$

which yields from (3.5) that

$$\vec{y}^* \le \ln K^+ + r^+ + \epsilon.$$

Letting $\epsilon \to 0$ in the above inequality leads to

$$\vec{y}^* \le \ln K^+ + r^+ := \alpha. \tag{3.8}$$

On the other hand, by Lemma 2.5, for $\forall \epsilon > 0$, $T_\epsilon^- \ne \emptyset$, that is, there is a point $\eta \in \mathbb{R}\backslash\mathbb{Z}$ such that

$$\vec{y}(\eta) \in [\vec{y}_*, \vec{y}_* + \epsilon] \quad \text{and} \quad \left.\frac{\mathrm{d}\,\vec{y}(t)}{\mathrm{d}\,t}\right|_{t=\eta} \le 0. \tag{3.9}$$

In view of (3.4), there exists $n_1 \in \mathbb{Z}$ such that $\eta \in (n_1, n_1+1)$ and

$$\frac{\mathrm{d}\,\vec{y}(t)}{\mathrm{d}\,t}\bigg|_{t=\eta} = \lambda r(n_1)\left[1 - \frac{e^{\vec{y}(n_1)}}{K(n_1)}\right],$$

which implies from (3.9) that

$$\vec{y}(n_1) \geq \ln K^-. \tag{3.10}$$

Further, it follows from (3.4) and (3.8) that

$$\int_{n_1}^{\eta} \frac{\mathrm{d}\,\vec{y}(t)}{\mathrm{d}\,t}\,\mathrm{d}t = \int_{n_1}^{\eta} \lambda r(n_1)\left[1 - \frac{e^{\vec{y}(n_1)}}{K(n_1)}\right]\mathrm{d}t \geq -\frac{r^+e^{\alpha}}{K^-}. \tag{3.11}$$

By (3.10)-(3.11), we have

$$\vec{y}(\eta) = \vec{y}(n_1) + \int_{n_1}^{\eta} \frac{\mathrm{d}\,\vec{y}(t)}{\mathrm{d}\,t}\,\mathrm{d}t \geq \ln K^- - \frac{r^+e^{\alpha}}{K^-},$$

which yields from (3.9) that

$$\vec{y}_* \geq \ln K^- - \frac{r^+e^{\alpha}}{K^-} - \epsilon.$$

Letting $\epsilon \to 0$ in the above inequality leads to

$$\vec{y}_* \geq \ln K^- - \frac{r^+e^{\alpha}}{K^-} := \beta. \tag{3.12}$$

From (3.8) and (3.12), it follows that

$$\beta \leq \vec{y}_* \leq \vec{y}^* \leq \alpha \Rightarrow e^{\beta} \leq \vec{x}_* \leq \vec{x}^* \leq e^{\alpha} \Rightarrow e^{\beta} \leq x_* \leq x^* \leq e^{\alpha} \Rightarrow \beta \leq y_* \leq y^* \leq \alpha.$$

Set $C = |\alpha| + |\beta| + 1$. Clearly, C is independent of $\lambda \in (0,1)$. Let $\Omega = \{y \in \mathbb{X} : \|y\|_{\mathbb{Z}} < C\}$. Therefore, Ω satisfies condition (a) of Lemma 3.1. Now we show that condition (b) of Lemma 3.1 holds, i.e., we prove that $QNy \neq 0$ for all $y \in \partial\Omega \cap \mathrm{Ker}L = \partial\Omega \cap \mathbb{R}$. If it is not true, then there exists at least one constant vector $y \in \partial\Omega \cap \mathbb{R}$ such that

$$0 = m\left[r(n)\left[1 - \frac{e^{y}}{K(n)}\right]\right].$$

So $\beta \leq y \leq \alpha$. Then $y \in \Omega \cap \mathbb{R}$. This contradicts the fact that $y \in \partial\Omega$. This proves that condition (b) of Lemma 3.1 holds. Finally, by a direct computation yields

$$\deg\big(JQN, \Omega \cap \mathrm{Ker}L, 0\big) = \deg\big(QN, \Omega \cap \mathrm{Ker}L, 0\big) = -1,$$

where $\deg(\cdot,\cdot,\cdot)$ is the Brouwer degree and J is the identity mapping since $\mathrm{Im}Q = \mathrm{Ker}L$. Obviously, all the conditions of Lemma 3.1 are satisfied. Therefore, Eq. (3.1) has at least one almost periodic sequence solution, that is, Eq. (1.1) has at least one positive almost periodic sequence solution. This completes the proof. □

Proof of Theorem 1.3. From Theorem 3.1, we know that Eq. (1.1) has at least one positive almost periodic sequence solution x. Suppose that y is another positive solution of Eq. (1.1).

It is not difficult to obtain that

$$x(n+1) = x(n)\exp\left(r(n)\left[1 - \frac{x(n)}{K(n)}\right]\right), \quad n \in \mathbb{Z}, \tag{3.13}$$

$$y(n+1) = y(n)\exp\left(r(n)\left[1 - \frac{y(n)}{K(n)}\right]\right), \quad n \in \mathbb{Z}. \tag{3.14}$$

Define

$$V(n) = |\ln x(n) - \ln y(n)|, \quad \forall t \in (n, n+1),\ n \in \mathbb{Z}.$$

Since $\vec{x}$ and $\vec{y}$ are continuous, then $V \in C(\mathbb{R})$.

By calculating the upper right derivative of V along (3.13)-(3.14), it follows that

$$D^+V(t) \leq -\frac{r^-}{K^+}|\vec{x}(n) - \vec{y}(n)|, \quad \forall t \in (n, n+1),\ n \in \mathbb{Z}.$$

Therefore, V is non-increasing. Sine $V \in C(\mathbb{R})$, integrating of the last inequality from 0 to $n \in \mathbb{Z}$ leads to

$$\begin{aligned} V(n) - V(0) &= \sum_{k=0}^{n-1}[V(k+1) - V(k)] \\ &= \sum_{k=0}^{n-1}\int_k^{k+1} D^+V(t)\,\mathrm{d}t \\ &\leq -\frac{r^-}{K^+}\sum_{k=0}^{n-1}|\vec{x}(k) - \vec{y}(k)|, \end{aligned}$$

that is,

$$\sum_{k=0}^{\infty}|\vec{x}(k) - \vec{y}(k)| < +\infty,$$

which implies that

$$\lim_{k\to+\infty}|\vec{x}(k) - \vec{y}(k)| = \lim_{k\to+\infty}|x(k) - y(k)| = 0.$$

The global asymptotical stability implies that the almost periodic sequence solution is unique. This completes the proof. □

4 An Example with a Numerical Simulation

The following example shows the feasibility of the main result of this paper.

Example 4.1. *Consider the following discrete logistic equation:*

$$x(n+1) = x(n)\exp\left((2 + \sin^4(\sqrt{2}t))\left[1 - \frac{x(n)}{1 + 0.1\cos(\sqrt{3}t)}\right]\right). \tag{4.1}$$

Clearly, Eq. (4.1) does not satisfy $\frac{K^+}{K^-}\exp(r^+-1) < 2$ *or* $r^+ < 2$*, which implies that it is impossible to ensure the existence of a unique globally asymptotically stable almost periodic solution by Theorems 1.1-1.2. However, by Theorem 1.3, one gets that Eq. (4.1) has a unique globally asymptotically stable almost periodic sequence solution (see Figure 1).*

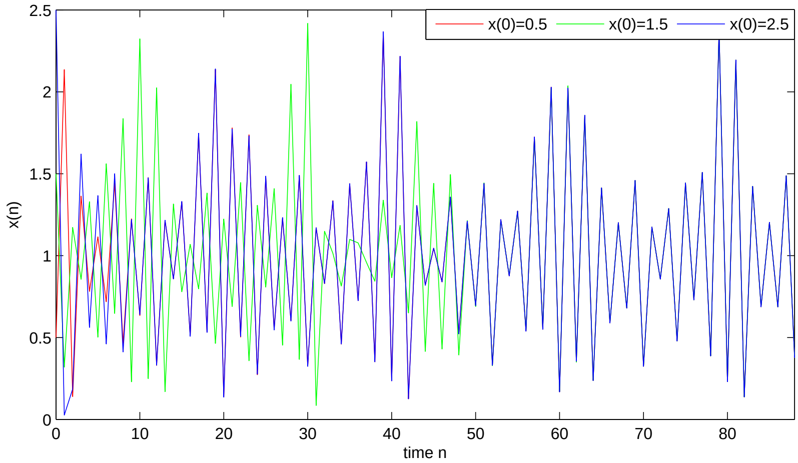

Fig. 1. Global asymptotical stability of almost periodic sequence x of Eq. (4.1)

5 Conclusions

The logistic models have been studied extensively, and many important phenomena have been observed in recent years. In this paper we study an almost periodic discrete logistic model. We obtain sufficient criteria for the existence and globally asymptotic stability of positive almost periodic solutions of the above model. In this paper, we only studied model without impulses. Whether model with impulses can be discussed in the same methods or not is still an open problem. We will continue to study this problem in the future.

Competing Interests

The author has declares that they have no competing interests.

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Appendix A. Proof of Some Lemmas

Define the fourier transform by

$$\hat{\varphi}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\mathrm{i}ut}\varphi(t)\,\mathrm{d}t.$$

In most cases, if φ have compact support and have enough smoothness, then the following inversion formula holds:

$$\varphi(t) = \int_{-\infty}^{\infty} e^{\mathrm{i}ut}\hat{\varphi}(u)\,\mathrm{d}u.$$

Lemma 5.1. *Assume that $f \in AP(\mathbb{Z})$ and $\tilde{f} \in AP(\mathbb{R})$ is defined as that in* (2.9)*. Let*

$$\tilde{f}(t) \sim \sum_{k=1}^{\infty} a_k e^{\mathrm{i}\lambda_k t}$$

with $\lambda_k \in \Theta = \bigcup_{l\in\mathbb{K}}[2l\pi+\rho, 2(l+1)\pi-\rho]$, where $\mathbb{K} = \{k_1, k_2, \ldots, k_n\} \subset \mathbb{Z}$, $k, n \in \mathbb{N}$ and $\rho \in (0, \frac{\pi}{2})$. Define

$$g(n) = \sum_{t=0}^{n-1} f(t) = \sum_{t=0}^{n-1} \tilde{f}(t), \quad n \in \mathbb{Z}.$$

Then $g \in AP(\mathbb{Z})$ and there exists a constant M independent of f and g such that $\|g\|_{\mathbb{Z}} \le M\|f\|_{\mathbb{Z}}$.

Proof. For some $\mathbb{K} = \{k_1, k_2, \ldots, k_n\} \subset \mathbb{Z}$, $n \in \mathbb{N}$ and $\rho \in (0, \frac{\pi}{2})$, as defined in the proof of Theorem 4.8 in [3, P_{67}], let $\varphi(t)$ be a function which is equal to $\frac{1}{e^{\mathrm{i}t}-1}$ when $t \in \Theta = \bigcup_{k\in\mathbb{K}}[2k\pi+\rho, 2(k+1)\pi-\rho]$, vanishes outside some finite interval $[-L, L]$ $(L > 0)$, and is in $C^{\infty}(\mathbb{R})$. Then

$$\begin{aligned} 2\pi|\hat{\varphi}(u)| &= \left|\int_{-\infty}^{\infty} e^{-\mathrm{i}ut}\varphi(t)\,\mathrm{d}t\right| = \left|\frac{1}{\mathrm{i}u}\int_{-\infty}^{\infty} e^{-\mathrm{i}ut}\varphi'(t)\,\mathrm{d}t\right| \\ &= \left|-\frac{1}{u^2}\int_{-\infty}^{\infty} e^{-\mathrm{i}ut}\varphi''(t)\,\mathrm{d}t\right| = \left|-\frac{1}{u^2}\int_{-L}^{L} e^{-\mathrm{i}ut}\varphi''(t)\,\mathrm{d}t\right| \\ &\le \frac{2LC_0}{u^2}, \quad C_0 := \sup_{s\in[-L,L]}\{|\varphi''(s)|\}. \end{aligned}$$

So $\hat{\varphi} \in L_1(\mathbb{R})$, there results that the inversion formula holds.

Let $\tilde{f}_0$ be a trigonometric polynomial $\tilde{f}_0(t) = \sum_{k=1}^{N} a_k e^{\mathrm{i}\lambda_k t}$ with $\lambda_k \in \Theta$, $k = 1, 2, \ldots, N$, N is a positive integer, $t \in \mathbb{R}$. Define $g_0(n) = \sum_{t=0}^{n-1} \tilde{f}_0(t) = \sum_{t=0}^{n-1}\sum_{k=1}^{N} a_k e^{\mathrm{i}\lambda_k t}$, $n \in \mathbb{Z}$. It follows that

$$\begin{aligned} g_0(n) &= \sum_{t=0}^{n-1}\sum_{k=1}^{N} a_k e^{\mathrm{i}\lambda_k t} = \sum_{k=1}^{N} a_k \frac{e^{\mathrm{i}\lambda_k n}-1}{e^{\mathrm{i}\lambda_k}-1} \\ &= \sum_{k=1}^{N} a_k\varphi(\lambda_k)e^{\mathrm{i}\lambda_k n} + \sum_{k=1}^{N} a_k\varphi(\lambda_k)e^{\mathrm{i}\lambda_k 0} \\ &= \sum_{k=1}^{N} a_k\int_{-\infty}^{\infty} e^{\mathrm{i}u\lambda_k}\hat{\varphi}(u)\,\mathrm{d}u\,e^{\mathrm{i}\lambda_k n} + \sum_{k=1}^{N} a_k\int_{-\infty}^{\infty} e^{\mathrm{i}u\lambda_k}\hat{\varphi}(u)\,\mathrm{d}u\,e^{\mathrm{i}\lambda_k 0} \\ &= \int_{-\infty}^{\infty}\sum_{k=1}^{N} a_k e^{\mathrm{i}(u+n)\lambda_k}\hat{\varphi}(u)\,\mathrm{d}u + \int_{-\infty}^{\infty}\sum_{k=1}^{N} a_k e^{\mathrm{i}u\lambda_k}\hat{\varphi}(u)\,\mathrm{d}u \\ &= \int_{-\infty}^{\infty}\tilde{f}_0(u+n)\hat{\varphi}(u)\,\mathrm{d}u + \int_{-\infty}^{\infty}\tilde{f}_0(u)\hat{\varphi}(u)\,\mathrm{d}u. \end{aligned}$$

Then

$$\|g_0\|_{\mathbb{Z}} \le 2\|\hat{\varphi}\|_1 \|\tilde{f}_0\|_{\mathbb{R}} := M\|\tilde{f}_0\|_{\mathbb{R}},$$

where $M := 2\|\hat{\varphi}\|_1$ and $\|\cdot\|_1$ is the norm of $L_1(\mathbb{R})$.

By Bochner's approximation theorem [7, P_{114}] (or [3, P_{48}]), there exists a sequence of trigonometric polynomials $\{\sigma_k(t); k \ge 1\}$ such that $\sigma_k(t) \to \tilde{f}(t)$ as $k \to \infty$ in $AP(\mathbb{R})$, with $\sigma_k(t)$ of the form

$$\sigma_k(t) = \sum_{l=1}^{m(N)} a_l^{(k)} e^{\mathrm{i}\lambda_l t}, \quad t \in \mathbb{R}.$$

In other words, the exponents of $\sigma_k(t)$ are chosen from those of $\tilde{f}$.

Let us now define

$$\check{\sigma}_k(n) = \sum_{t=0}^{n-1} \sigma_k(t) = \sum_{t=0}^{n-1} \sum_{l=1}^{m(N)} a_l^{(k)} e^{\mathrm{i}\lambda_l t}, \quad n \in \mathbb{Z}.$$

Similar to the proof of Theorem 5.2 in [7, P_{141}], ones could get that $g(n) = \lim_{k\to\infty} \check{\sigma}_k(n) = \sum_{t=0}^{n-1} \tilde{f}(n) \in AP(\mathbb{Z})$ and

$$\|g\|_{\mathbb{Z}} \le M\|\tilde{f}\|_{\mathbb{R}} = M\|f\|_{\mathbb{Z}}.$$

This completes the proof. □

Lemma 5.2. *$\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces endowed with $\|\cdot\|_{\mathbb{Z}}$.*

Proof. Assume that $y_k \in \mathbb{V}$ and $\lim_{k\to\infty} y_k = y_0$. Then

$$\begin{aligned} \lim_{k\to\infty} \tilde{y}_k(t) &= \lim_{k\to\infty} \big\{[y_k(n+1) - y_k(n)](t-n) + y_k(n)\big\} \\ &= [y_0(n+1) - y_0(n)](t-n) + y_0(n) \\ &= \tilde{y}_0(t), \quad \forall t \in [n, n+1),\ n \in \mathbb{Z}. \end{aligned}$$

Since $y_k \in \mathbb{V}$, for all $\theta \notin \Theta$, ones have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \tilde{y}_k(s) e^{-\mathrm{i}\theta s}\, \mathrm{d}s = 0, \quad k = 1, 2, \ldots.$$

Thus

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \tilde{y}_0(s) e^{-\mathrm{i}\theta s}\, \mathrm{d}s = 0,$$

which implies that $\forall \theta \in \Lambda(y_0), \theta \in \Theta$. It is easy to see that $\mathbb{V}$ is a Banach space endowed with $\|\cdot\|_{\mathbb{Z}}$. The same can be concluded for $\mathbb{X}$ and $\mathbb{Y}$. This completes the proof. □

Lemma 5.3. *L defined in Theorem* 3.1 *is a Fredholm mapping of index zero.*

Proof. It is obvious that L is a linear operator and $\mathrm{Ker}L = \mathbb{R}$. It remains to prove that $\mathrm{Im}L = \mathbb{V}$. Suppose that $\phi \in \mathrm{Im}L \subseteq \mathbb{Y}$, there exist $\phi_1 \in \mathbb{V}$ and $\phi_2 \in \mathbb{R}$ such that $\phi = \phi_1 + \phi_2$. By Lemma 5.1, we have $\sum_{k=0}^{n-1} \phi_1(k) \in AP(\mathbb{Z})$. Since $\phi \in \mathrm{Im}L$, there exists $\upsilon \in \mathbb{X}$ such that $L\upsilon = \Delta\upsilon = \phi$, which implies that

$$\left|\sum_{k=0}^{n-1} \phi(k)\right| = \left|\sum_{k=0}^{n-1} \Delta\upsilon(k)\right| \le |\upsilon(n) - \upsilon(0)| < +\infty, \quad \forall n \in \mathbb{Z}.$$

Then

$$|\phi_2||n| = \left|\sum_{k=0}^{n-1} \phi_2\right| = \left|\sum_{k=0}^{n-1} \big[\phi(k) - \phi_1(k)\big]\right| < +\infty, \quad \forall n \in \mathbb{Z},$$

which implies that $\phi_2 = 0$. Therefore, $\phi = \phi_1 \in \mathbb{V}$. This tells us that $\mathrm{Im}L \subseteq \mathbb{V}$.

In the following, we will prove that $\mathbb{V} \subseteq \mathrm{Im}L$. Suppose that $\varphi \in \mathbb{V}$, by Lemma 5.1, $\Phi(n) = \sum_{k=0}^{n-1} \varphi(k) \in AP(\mathbb{Z})$. Indeed, if $\theta \in \Theta$, let $\chi(\theta) = 1 - \frac{1}{\mathrm{i}\theta e^{\mathrm{i}\theta}} - \frac{1}{e^{\mathrm{i}\theta}}$, then we obtain

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{T}\int_0^T \tilde{\Phi}(t)e^{-\mathrm{i}\theta t}\,\mathrm{d}t &= \lim_{n\to\infty} \frac{1}{n}\int_0^n \tilde{\Phi}(t)e^{-\mathrm{i}\theta t}\,\mathrm{d}t \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\int_k^{k+1} \big[(\Phi(k+1)-\Phi(k))(t-k)+\Phi(k)\big]e^{-\mathrm{i}\theta t}\,\mathrm{d}t \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}(1-\frac{1}{\mathrm{i}\theta e^{\mathrm{i}\theta}}-\frac{1}{e^{\mathrm{i}\theta}})(\Phi(k+1)-\Phi(k))e^{-\mathrm{i}\theta k} \\
&\quad + \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\mathrm{i}\theta}(1-\frac{1}{e^{\mathrm{i}\theta}})\Phi(k)e^{-\mathrm{i}\theta k} \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\chi(\theta)e^{\mathrm{i}\theta}\Phi(k+1)e^{-\mathrm{i}\theta(k+1)} \\
&\quad - \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\chi(\theta)\Phi(k)e^{-\mathrm{i}\theta k} + \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\mathrm{i}\theta}(1-\frac{1}{e^{\mathrm{i}\theta}})\Phi(k)e^{-\mathrm{i}\theta k} \\
&= \left[(e^{\mathrm{i}\theta}-1)\chi(\theta)+\frac{1}{\mathrm{i}\theta}(1-\frac{1}{e^{\mathrm{i}\theta}})\right]\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\Phi(k)e^{-\mathrm{i}\theta k} \\
&= e^{-\mathrm{i}\theta}(e^{\mathrm{i}\theta}-1)^2\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\Phi(k)e^{-\mathrm{i}\theta k} \\
&= e^{-\mathrm{i}\theta}(e^{\mathrm{i}\theta}-1)^2\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\sum_{l=0}^{k-1}\varphi(l)e^{-\mathrm{i}\theta k} \\
&= e^{-\mathrm{i}\theta}(e^{\mathrm{i}\theta}-1)\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\varphi(k)e^{-\mathrm{i}\theta k} \\
&= (e^{\mathrm{i}\theta}-1)\lim_{T\to\infty} \frac{1}{T}\int_0^T \tilde{\varphi}(t)e^{-\mathrm{i}\theta t}\,\mathrm{d}t.
\end{aligned}$$

Let $\psi(n) = \Phi(n) - m\big(\Phi(n)\big)$, $\forall n \in \mathbb{Z}$. So $\Lambda(\tilde{\psi}) = \Lambda(\tilde{\varphi})$. Therefore, $\psi \in \mathbb{V} \subseteq \mathbb{X}$. Further, we have

$$\Delta\psi(n) = \Delta\big[\Phi(n) - m\big(\Phi(n)\big)\big] = \varphi(n), \quad \forall n \in \mathbb{Z},$$

which implies that $\varphi \in \mathrm{Im}L$. Hence, we deduce that $\mathbb{V} \subseteq \mathrm{Im}L$. Therefore, $\mathbb{V} = \mathrm{Im}L$.

Furthermore, one can easily show that $\mathrm{Im}L$ is closed in $\mathbb{Y}$ and

$$\dim \mathrm{Ker}L = 1 = \mathrm{codim}\,\mathrm{Im}L.$$

Therefore, L is a Fredholm operator of index zero. This completes the proof. □

Lemma 5.4. *N defined in Theorem 3.1 is L-compact on $\bar{\Omega}$(Ω is an open-bounded subset of $\mathbb{X}$).*

Proof. Let P and Q are defined as that in Theorem 3.1. Obviously, P and Q are continuous such that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q$. Further, we have $(I-Q)\mathbb{R} = \{0\}$ and $(I-Q)\mathbb{V} = \mathbb{V}$. Hence, $\mathrm{Im}(I-Q) = \mathbb{V} = \mathrm{Im}L$. In view of $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I-Q)$, through an easy computation we find that the inverse $K_P : \mathrm{Im}L \to \mathrm{Ker}P \cap \mathrm{Dom}L$ of L_P exists and is given in

Theorem 3.1.

Furthermore, QN and $(I - Q)N$ defined in Theorem 3.1 are continuous. We claim that K_P is also continuous. Assume that $y_n \in \mathrm{Im}L = \mathbb{V}\,(n \in \mathbb{N})$ such that

$$\lim_{n\to\infty} y_n = y_0.$$

By the completeness of $\mathbb{V}$, $y_0 \in \mathbb{V}$ and $y_n - y_0 \in \mathbb{V}\,(n \in \mathbb{N})$. Then we have from Lemma 5.1 that

$$|K_P y_n - K_P y_0|_\infty \leq 2M|y_n - y_0|_\infty, \quad n \in \mathbb{N}.$$

Therefore, $\lim_{n\to\infty} |K_P y_n - K_P y_0|_\infty = 0$. So K_P and $K_P(I - Q)$ are also continuous. In addition, $K_P(I - Q)z$ are uniformly bounded on $\bar{\Omega}$. It is not difficult to verify that $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)Nz$ is equicontinuous on $\bar{\Omega}$. Hence, by the Arzela-Ascoli theorem, we can conclude that $K_P(I - Q)N(\bar{\Omega})$ is compact. Thus N is L-compact on $\bar{\Omega}$. This completes the proof. □

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