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Analytical and Numerical Solutions for a Rotating Annular Disk of Variable Thickness

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Abstract

In this paper, the analytical and numerical solutions for rotating variable-thickness solid disk and numerical solution for rotating variable-thickness annular disk are presented. The outer edge of the solid disk and the inner and outer edges of the annular disk are considered to have clamped boundary conditions. Two different cases for the radially varying thickness of the solid and annular disks are given. The numerical solution as well as the analytical solution is available for the first case of the solid disk while the analytical solution is not available for the second case of the annular disk. Both analytical and numerical results for displacement and stresses will be investigated for the first case of radially varying thickness. The accuracy of the present numerical solution is discussed and its ability of use for the second case of radially varying thickness is investigated. Finally, the distributions of displacement and stresses will be presented and the appropriate comparisons and discussions are made at the same angular velocity.

Keywords: Rotating, Annular Disk, Solid Disk, Finite Difference Method

1. Introduction

The problems of rotating solid and annular disks have been performed under various interesting assumptions and the topic can be easily found in most of the standard elasticity books [1-5]. Most of the research works are concentrated on the analytical solutions of rotating disks with simple cross-section geometries of uniform thickness and especially variable thickness [6-14]. The analytical elasticity solutions of such rotating disks are available in many books of elasticity.

As many rotating components in use have complex cross-sectional geometries, they cannot be dealt with using the existing analytical methods. Numerical methods, such as the finite element method [15], the boundary element method [16] and Runge-Kutta's algorithm [17], can be applied to cope with these rotating components.

In this paper, we will present the analytical solution for the rotating solid disk with arbitrary cross-section of continuously variable thickness. In the following, a unified governing equation will be first derived from the basic equations of the rotating disks and the proposed stress-strain relationship. Next, finite difference method (FDM) is introduced to solve the governing equation. A comparison between both analytical and numerical solutions is made. The accuracy of the numerical solution is used to find the displacement and stresses of rotating variable-thickness annular disk whose analytical solution is not available. Finally, a number of numerical examples are given to demonstrate the validity of the proposed method.

2. Basic Equations

As the effect of thickness variation of rotating disks can be taken into account in their equation of motion, the theory of the variable-thickness disks can give good results as that of the uniform-thickness disks as long as they meet the assumption of plane stress. After considering this effect, the equation of motion of rotating disks with variable thickness can be written as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(hr\sigma_r\right)-h\sigma_\theta+h\rho\omega^2r^2=0, \qquad (1)$$

where σ_r and σ_θ are the radial and circumferential stresses, r is the radial coordinate, ρ is the density of the rotating disk, ω is the constant angular velocity, and h is the thickness which is function of the radial coordinate r.

The relations between the radial displacement u_r and the strains are irrespective of the thickness of the rotating disk. They can be written as

$$\varepsilon_r = \frac{\mathrm{d}u_r}{\mathrm{d}r}, \ \varepsilon_\theta = \frac{u_r}{r}, \tag{2}$$

where ε_r and ε_θ are the radial and circumferential strains, respectively.

For the elastic deformation, the constitutive equations for the rotating disk can be described with Hooke's law

$$\varepsilon_r = \frac{\sigma_r - \nu\sigma_\theta}{E}, \ \varepsilon_\theta = \frac{\sigma_\theta - \nu\sigma_r}{E}. \tag{3}$$

Using (2) into Equation (3), one can obtain the constitutive equations for σ_r and σ_θ as:

$$\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}\left(\frac{\mathrm{d}u_r}{\mathrm{d}r} + \nu\frac{u_r}{r}\right), \\ \sigma_\theta &= \frac{E}{1-\nu^2}\left(\frac{u_r}{r} + \nu\frac{\mathrm{d}u_r}{\mathrm{d}r}\right). \end{aligned} \tag{4}$$

Let us consider a symmetric thin disk with respect to the mid-plane, its profile varying in the radial direction according to the formula see **Figure 1**:

$$h(r) = h_0\left[1 - n\left(\frac{r}{b}\right)^k\right], \tag{5}$$

where h_0 is the thickness at the axis of the disk, n and k are geometric parameters $h(r) = (0 \le n < 1,\ k > 0)$, and b is the outer radius of the solid disk. A uniform-thickness disk is obtained by setting $n = 0$ and a linearly decreasing thickness is obtained by setting $k = 1$. Furthermore, if $k < 1$, the profile is concave and if $k > 1$, it is convex. The thickness of the disk is assumed to be sufficiently small compared to its diameter so that

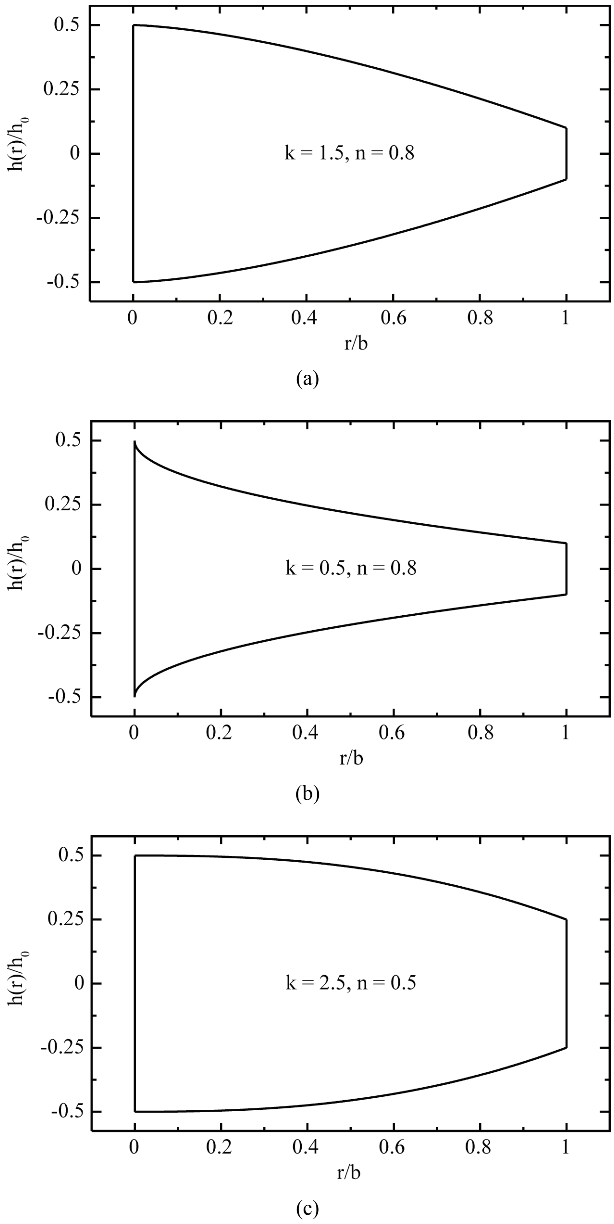


Figure 1. Variable-thickness solid disk profiles for (a) k = 1.5 and n = 0.8; (b) k = 0.5 and n = 0.8 and (c) k = 2.5 and n = 0.5.

$$\begin{aligned} &R = r/b \\ &\Omega = \omega b\sqrt{\rho\left(1-\nu^2\right)}, \\ &U(R) = \frac{E}{b\Omega^2}u_r(r), \\ &\left(\bar{\varepsilon}_r, \bar{\varepsilon}_\theta\right) = \frac{E}{\Omega^2}\left(\varepsilon_r, \varepsilon_\theta\right), \\ &\left(\bar{\sigma}_r, \bar{\sigma}_\theta\right) = \frac{1-\nu^2}{\Omega^2}\left(\sigma_r, \sigma_\theta\right). \end{aligned} \tag{7}$$

3. Formulation and Elastic Solution for Solid Disk

The substitution of Equations (4) and (5) into Equation (1) produces the following confluent hypergeometric differential equation for the radial displacement $u_r(r)$:

$$\begin{aligned} &r^2\frac{\mathrm{d}^2u_r}{\mathrm{d}r^2} + \frac{r}{1-n\left(\frac{r}{b}\right)^k}\left[1 - n(1+k)\left(\frac{r}{b}\right)^k\right]\frac{\mathrm{d}u_r}{\mathrm{d}r} \\ &-\left[1 + \frac{\nu kn\left(\frac{r}{b}\right)^k}{1-n\left(\frac{r}{b}\right)^k}\right]u_r + \frac{1-\nu^2}{E}\rho\omega^2r^3 = 0. \end{aligned} \tag{6}$$

Then, Equation (6) may be written in the following simple form

$$R^2\frac{\mathrm{d}^2U}{\mathrm{d}R^2}+\frac{\left[1-n\left(1+k\right)R^k\right]R}{1-nR^k}\frac{\mathrm{d}U}{\mathrm{d}R} -\frac{1-n\left(1-\nu k\right)R^k}{1-nR^k}U+R^3=0. \quad (8)$$

The general solution of the above equation can be written as

$$U\left(R\right)=C_1F_1\left(R\right)+C_2F_2\left(R\right)+P\left(R\right), \quad (9)$$

where C_1 and C_2 are arbitrary constants and F_1 and F_2 are given by:

$$F_1\left(R\right)=RH\left(\left[\alpha,\beta\right],\left[\gamma\right],nR^k\right), \quad (10)$$

$$F_2\left(R\right)=\frac{1}{R}H\left(\left[1-\alpha,1-\beta\right],\left[2-\gamma\right],nR^k\right), \quad (11)$$

in which

$$\alpha=\frac{1}{2}+\frac{1}{k}-\frac{\sqrt{k^2+4\left(1-k\nu\right)}}{2k}, \quad \beta=\frac{1}{2}+\frac{1}{k}+\frac{\sqrt{k^2+4\left(1-k\nu\right)}}{2k},\ \gamma=1+\frac{2}{k}. \quad (12)$$

The functions $H([\xi,\eta],[\zeta],z)$ are the generalized hyper-geometric functions,

$$H\left(\left[\xi,\eta\right],\left[\zeta\right],z\right)=\sum_{q=0}^{\infty}\frac{\left(\xi\right)_q\left(\eta\right)_q}{\left(\zeta\right)_q q!}z^q, \quad \zeta\neq 0,\quad \left|z\right|<1, \quad (13)$$

where $(\lambda)_q$ is the Pochhammer symbol given by

$$\left(\lambda\right)_q=\lambda\left(\lambda+1\right)\left(\lambda+2\right)\ldots\left(\lambda+q-1\right)=\frac{\Gamma\left(\lambda+q\right)}{\Gamma\left(\lambda\right)}, \quad (14)$$

in which Γ represents Gamma function.

The term $P(R)$ in Equation (9) is the particular solution of Equation (8) which can be written as

$$P\left(R\right)=\left(4-k^2\right)\left(F_1\left(R\right)\int\frac{R^2F_2\left(R\right)}{\Delta}\mathrm{d}R-F_2\left(R\right)\int\frac{R^2F_1\left(R\right)}{\Delta}\mathrm{d}R\right), \quad (15)$$

where

$$\Delta=nk\left(1-\nu\right)\left(2+k\right)R^kF_1F_4-\left(2-k\right)F_2\left[2\left(2+k\right)F_1+nk\left(1+\nu\right)R^kF_3\right], \quad (16)$$

in which

$$F_3\left(R\right)=RH\left(\left[1+\beta,1+\alpha\right],\left[1+\gamma\right],nR^k\right), \quad (17)$$

$$F_4\left(R\right)=\frac{1}{R}H\left(\left[2-\alpha,2-\beta\right],\left[3-\gamma\right],nR^k\right), \quad (18)$$

The substitution of Equation (9) into Equation (4) with the aid of the dimensionless forms given in Equation (7) gives the radial and circumferential stresses in the forms of (19) and (20):

$$\bar{\sigma}_r\left(R\right)=C_1\left(\frac{1+\nu}{R}F_1+\frac{\alpha\beta k^2nR^{k-1}}{2+k}F_3\right)-C_2\left(\frac{1-\nu}{R}F_2+\frac{\left(1-\alpha\right)\left(1-\beta\right)k^2nR^{k-1}}{2-k}F_4\right)+\frac{\mathrm{d}P}{\mathrm{d}R}+\frac{\nu P}{R}, \quad (19)$$

$$\bar{\sigma}_\theta\left(R\right)=C_1\left(\frac{1+\nu}{R}F_1+\frac{\nu\alpha\beta k^2nR^{k-1}}{2+k}F_3\right)+C_2\left(\frac{1-\nu}{R}F_2-\frac{\nu\left(1-\alpha\right)\left(1-\beta\right)k^2nR^{k-1}}{2-k}F_4\right)+\nu\frac{\mathrm{d}P}{\mathrm{d}R}+\frac{P}{R}. \quad (20)$$

4. Analytical Solution for the Rotating Solid Disk

The analytical elastic solution for the solid disk with variable-thickness is completed by the application of the boundary conditions. Since the radial displacement should be vanished and the stresses should be finite at the center of the disk, then the constant C_2 vanishes. The radial displacement is vanished at the outer edge of the disk, $r = b$ or ($R = 1$), hence

$$C_1=-\frac{P\left(1\right)}{F_1\left(1\right)}. \quad (21)$$

So, one can easily obtain the solution for the present rotating variable-thickness solid disk by the substitution of Equation (21) into Equations (9), (19) and (20).

5. Finite Difference Algorithm for Solid Disk

The resolution of the elastic problem of rotating disk with variable thickness is to solve a second-order differential equation, Equation (8), under the given boundary conditions. This equation can be written in the following general form:

$$U''=p\left(R\right)U'+q\left(R\right)U+s\left(R\right), \quad 0<R\leq 1,\ U\left(0\right)=U\left(1\right)=0, \quad (22)$$

where the prime (') denotes differentiation with respect to R and

$$p(R) = -\frac{1-n(1+k)R^k}{(1-nR^k)R},$$
$$q(R) = \frac{1-n(1-\nu k)R^k}{(1-nR^k)R^2}, \quad (23)$$
$$s(R) = -R.$$

It is clear that the above problem has a unique solution because $p(R)$, $q(R)$, and $s(R)$ are continuous on the given interval]0,1] and $q(R) > 0$ on]0,1]. The linear second-order boundary value problem given in Equation (22) requires that difference-quotient approximations be used for approximating U' and U''. First we select an integer $N > 0$ and divided the interval]0,1] into $(N+1)$ equal subintervals, whose end points are the mesh points $R_i = i\,\Delta R$, for $i = 0,1,...,N+1$, where $\Delta R = 1/(N+1)$. At the interior mesh points, R_i, $i = 1,2,...,N$, the differential equation to the approximated is

$$U''(R_i) = p(R_i)\,U'(R_i) + q(R_i)U(R_i) + s(R_i). \quad (24)$$

If we apply the centered difference approximations of $U'(R_i)$ and $U''(R_i)$ to Equation (24), we arrive at the system:

$$-\left[1+\frac{\Delta R}{2}p(R_i)\right]U_{i-1} + \left[2+(\Delta R)^2 q(R_i)\right]U_i - \left[1-\frac{\Delta R}{2}p(R_i)\right]U_{i+1} = -(\Delta R)^2 s(R_i), \quad (25)$$

for each $i = 1,2,...,N$. The N equations, together with the boundary conditions

$$U_0 = 0, \quad U_{N+1} = 0, \quad (26)$$

Are sufficient to determine the unknowns U_i, $i = 0,1,2,...,N+1$. The resulting system of Equations (25) is expresses in the tri-diagonal $N \times N$-matrix form:

$$AU = B, \quad (27)$$

where

$$A_{i,i} = -2-(\Delta R)^2 q(R_i),\ i = 1,2,...,N,$$
$$A_{i,i+1} = 1-\frac{\Delta R}{2}p(R_i),\ i = 1,2,...,N-1,$$
$$A_{i,i-1} = 1+\frac{\Delta R}{2}p(R_i),\ i = 2,3,...,N, \quad (28)$$
$$A_{i,j} = A_{j,i} = 0,\ \ i = 1,2,...,N-2,,\quad j = 3,4,...,N,\ j \ge i+2,$$
$$B_i = (\Delta R)^2 s(R_i),\ \ i = 1,2,...,N.$$

The solution of the finite difference discretization of the two-point linear boundary value problem can therefore be found easily even for very small mesh sizes.

6. Numerical Examples and Discussion for Solid Disk

Some numerical examples for the rotating variable-thickness solid disks will be given according the analytical and numerical solutions $(\nu = 0.3)$. According to Equation (7), the following dimensionless response characteristics

$$u = U(R) = \frac{E}{b\Omega^2}u_r(r),$$
$$\{\sigma_1,\sigma_2\} = \{\bar{\sigma}_r(R),\bar{\sigma}_\theta(R)\} = \frac{1-\nu^2}{\Omega^2}\{\sigma_r(r),\sigma_\theta(r)\}. \quad (29)$$

determined as per the analytical solution are compared with those obtained by the numerical FDM solution.

The results of the present investigations for the radial displacement u are reported in **Table 1**. For this example, N = 9, 19, 39 and 79, so ΔR has the corresponding values 0.1, 0.05, 0.025 and 0.0125, respectively.

If we use the Richardson extrapolation method with $\Delta R = 0.1, 0.05, 0.025,$ and 0.0125 we obtain results listed in **Table 2**. The first extrapolation is

$$\mathrm{Ext}_{1i} = \frac{4U_i(\Delta R = 0.05) - U_i(\Delta R = 0.1)}{3}; \quad (30)$$

the second extrapolation is

$$\mathrm{Ext}_{2i} = \frac{4U_i(\Delta R = 0.025) - U_i(\Delta R = 0.05)}{3}; \quad (31)$$

the third extrapolation is

$$\mathrm{Ext}_{3i} = \frac{4U_i(\Delta R = 0.0125) - U_i(\Delta R = 0.025)}{3}; \quad (32)$$

the forth extrapolation is

$$\mathrm{Ext}_{4i} = \frac{16\,\mathrm{Ext}_{2i} - \mathrm{Ext}_{1i}}{15}; \quad (33)$$

and the final extrapolation is

$$\mathrm{Ext}_{5i} = \frac{16\,\mathrm{Ext}_{3i} - \mathrm{Ext}_{2i}}{15}. \quad (34)$$

All of the results of Ext_{4i} and Ext_{5i} are correct to the decimal places listed. In fact, if sufficient digits are maintained, these approximations give results that agree with the exact solution with maximum error of 2.9×10^{-8} and 4.8×10^{-8}, respectively at the mesh points.

The distribution of the radial displacement, radial and circumferential stresses are presented in **Figure 2**. The numerical FDM solution is compared with the analytical solution for the rotating variable-thickness sold disk with k = 3 and various values of n. It is clear that, the FDM gives displacement and stresses with excellent accuracy with the exact analytical solution. It can be seen from **Figure 2** that the FDM can describe the displacement and stresses through-the-thickness very well enough.

7. Formulation and Numerical Solution for Annular Disk

Here we consider a thin annular disk varies continuously in the form of a form of a general parabolic function (see **Figure 3**):

$$h(r) = h_0\left[1 - n\left(\frac{r-a}{b-a}\right)^k\right], \tag{35}$$

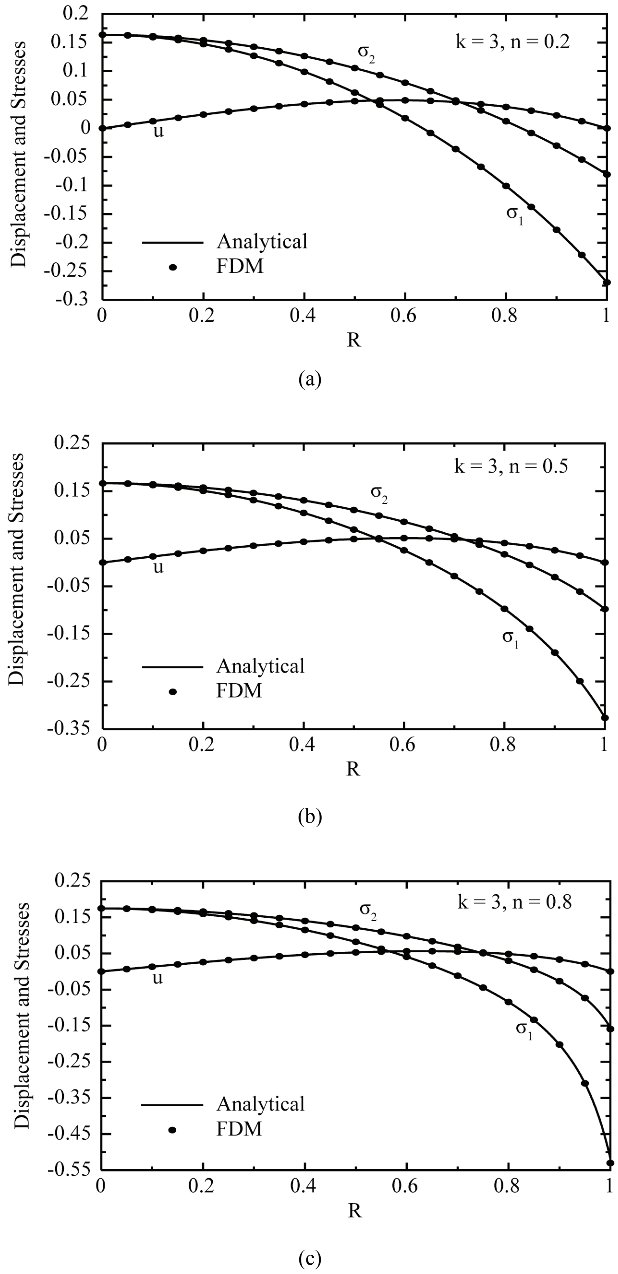


Figure 2. Dimensionless radial displacement *u*, radial stress σ_1 and circumferential stress σ_2 for the variable-thickness solid disk (*k* = 3): (a) *n* = 0.2, (b) *n* = 0.5 and (c) *n* = 0.8.

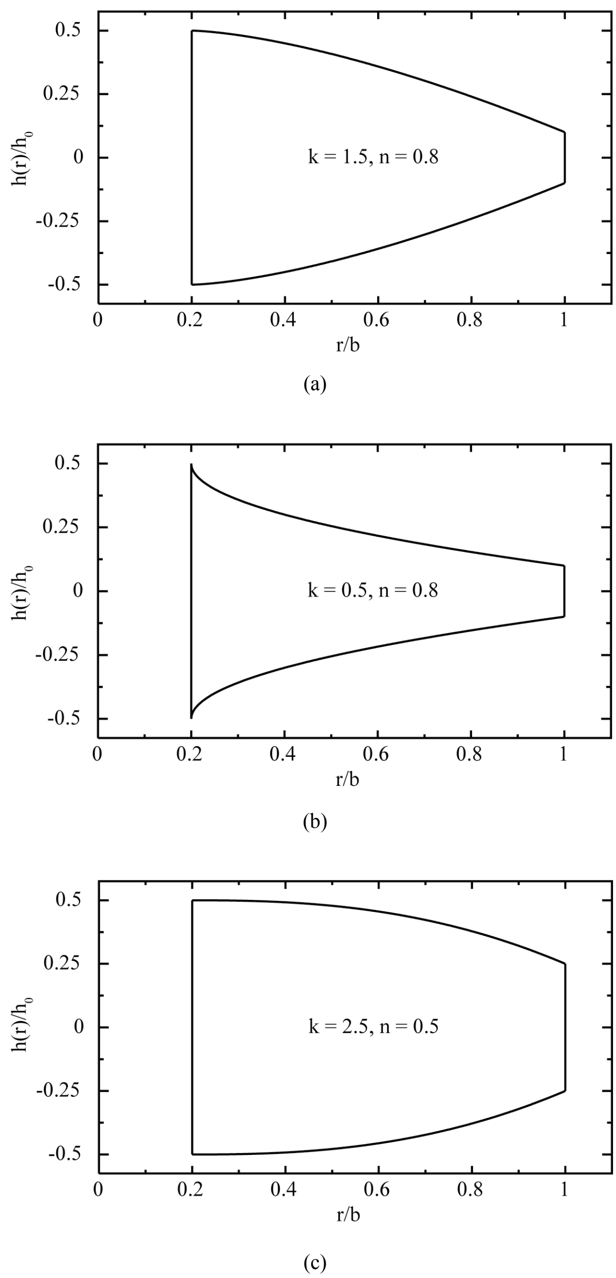
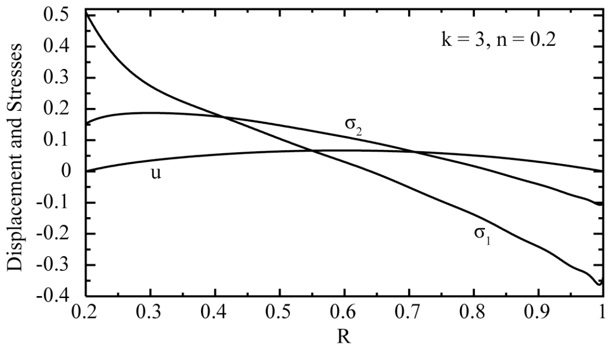


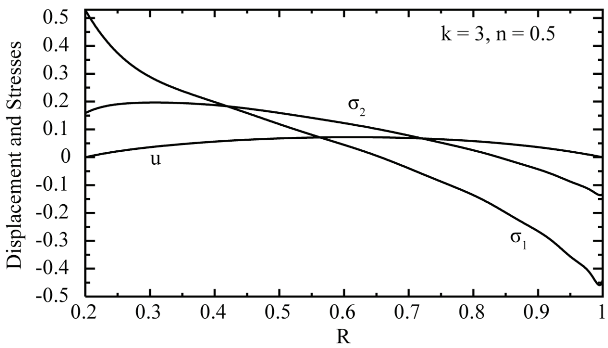
Figure 3. Variable-thickness annular disk profiles for (a) *k* = 1.5 and *n* = 0.8, (b) *k* = 0.5 and *n* = 0.8 and (c) *k* = 2.5 and *n* = 0.5.

where a is the inner radius of the annular disk. The equilibrium equation corresponding to Equation (35) may be easily given but its analytical solution is not.

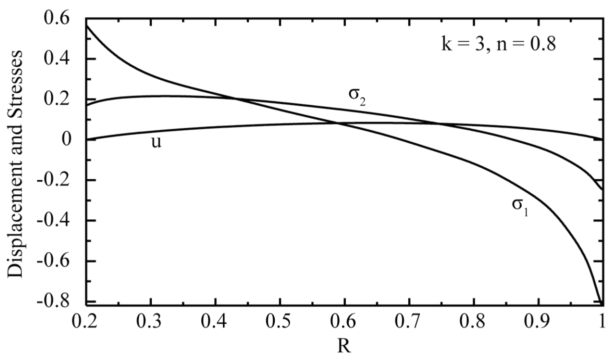
The substitution of Equations (4) and (35) into Equation (1) with the help of the dimensionless forms given in Equation (7), produces the following confluent hypergeometric differential equation for the radial displacement $U(R)$ of the annular disk:



(a)



(b)



(b)

Figure 4. Dimensionless radial displacement *u*, radial stress σ_1 and circumferential stress σ_2 for the variable-thickness annular disk (*k* = 3): (a) *n* = 0.2, (b) *n* = 0.5 and (c) *n* = 0.8.

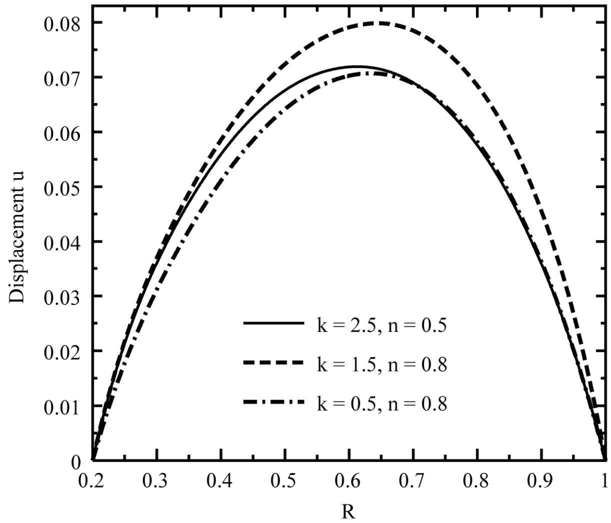


Figure 5. Dimensionless radial displacement *u* of the variable-thickness annular disk for di®erent values of *k* and *n*.

$$R^2\frac{\mathrm{d}^2U}{\mathrm{d}R^2}+R\left[1-\frac{nkR\bar{R}^k}{(R-A)\left(1-n\bar{R}^k\right)}\right]\frac{\mathrm{d}U}{\mathrm{d}R}-\left[1+\frac{n\nu kR\bar{R}^k}{(R-A)\left(1-n\bar{R}^k\right)}\right]U+\frac{R^4}{R-A}=0, \tag{36}$$

where

$$A=\frac{a}{b},\ \bar{R}=\frac{R-A}{1-A}. \tag{37}$$

Making analogous steps as given for the solid disk, Equation (36) can be written in the following general form:

$$U''=p(R)\,U'+q(R)U+s(R),\quad A<R\le 1,\ U(A)=U(1)=0, \tag{38}$$

where the prime $(')$ denotes differentiation with respect to R and

$$p(R)=-\frac{1}{R}\left[1-\frac{nkR\bar{R}^k}{(R-A)\left(1-n\bar{R}^k\right)}\right],\quad q(R)=\frac{1}{R^2}\left[1-\frac{n\nu kR\bar{R}^k}{(R-A)\left(1-n\bar{R}^k\right)}\right],\quad s(R)=-\frac{R^2}{R-A}. \tag{39}$$

So, FDM gives easily the radial displacement of the rotating variable-thickness annular disk. Using the curve fitting and least square method, one can obtain the radial and circumferential stresses. Taking n = 0.2, 0.5 and 0.8, and k = 0.5, 1.5, 2.5 and 3 in the variable thickness function given in Equation (35), a rotating annular disk with such variable thickness is studied. The inner and outer radii of the disk are taken to be $a = 0.2\ b$ ($R = A = 0.2$) and b ($R = 1$), and the results are given in terms of the rotating angular velocity. The results calculated with the FDM for displacement and stresses of a rotating variable-thickness annular disk are given in **Figures 3-7**.

8. Conclusions

This paper presents a unified numerical method for the elastic calculation of rotating disks with a general, arbitrary configuration. The governing equation was derived from the equilibrium equation and the stress- strain relationship. The analytical solution was given for the rotating variable-thickness solid disk. The calculation of the rotating sold and annular disks was turned into finding the solution of a second-order differential equation under the given conditions at two boundary fixed points. Finite difference method algorithm was introduced to solve the governing equation for both solid and annular disks and a number of numerical examples were studied. The results from the analytical and FDM solutions were compared. The proposed FDM approach gives very agreeable results to the analytical solution.

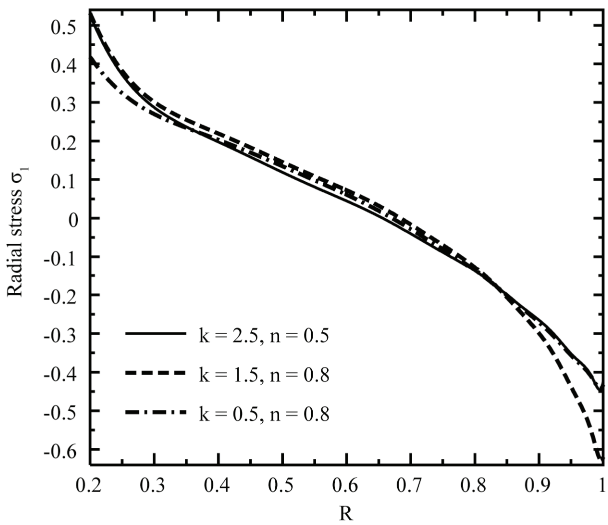


Figure 6. Dimensionless radial stress σ_1 in the variable-thickness annular disk for different values of *k* and *n*.

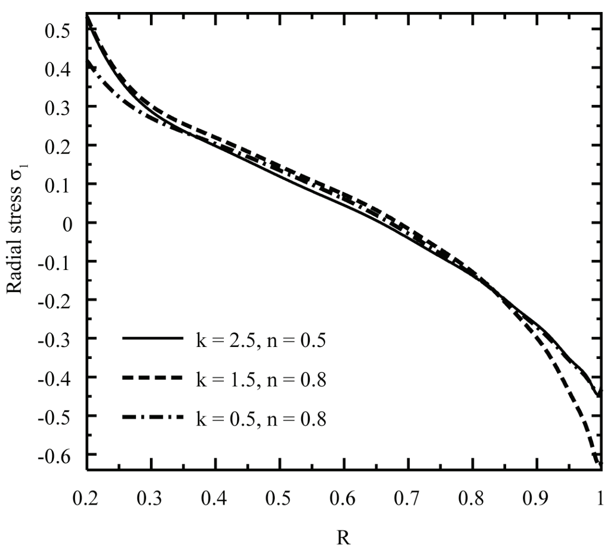


Figure 7. Dimensionless circumferential stress σ_2 in the variable-thickness annular disk for di®erent values of *k* and *n*.

9. Acknowledgements

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