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Characterization of Time to Failure in Prognostics: Brief Tutorial Guide to Prognostics Professionals

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Authors' contributions

This work was carried out in collaboration between the two authors. Author AMR designed the study, wrote the first draft of the manuscript and managed literature survey. Author HAB managed the analysis, implemented supporting algorithms, drew the figures, and contributed to literature survey. Both authors read and approved the final manuscript.

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Abstract

As a random variable, the survival time or Time to Failure (TTF) of a certain component or system can be fully characterized by its probability density function (pdf) $f_T(t)$ or its Cumulative Distribution Function (CDF) $F_T(t)$. Moreover, it might be also identified by transform functions such as the Moment Generating Function (MGF) and the Characteristic Function (CF). In reliability engineering, additional specific equivalent characterizations are used including the reliability function (survival function) which is the Complementary Cumulative Distribution Function (CCDF), and the failure rate (hazard rate), which is the probability density function normalized w.r.t. reliability. In prognostics, a prominent emerging subfield of reliability engineering, the characterizing functions are still supplemented by other specifically tailored ones. Notable among these is the Mean Residual Life (MRL) (also know as the Remaining Useful Life (RUL)). The purpose of this paper is to compile and interrelate the most prominent among these

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characterizing functions and explore their important properties. The paper points out that there is currently a significant proliferation of characterizing functions emerged in various fields. It shows that, under mild conditions, the product and quotient of two characterizing functions are also characterizing functions. The choice of one characterizing function in a certain application is a matter of convenience and taste. Our survey is far from being a conclusive one as it is intended to be just a brief tutorial guide for prognostics scholars, especially beginners. We had to arbitrarily leave out many of the less known characterizing functions such as the aging intensity function, log-odds rate, and entropy-related functions.

Keywords: Prognostics; Remaining Useful Life (RUL); Mean Residual Life (MRL); characterizing functions; hazard rate; reliability.

Nomenclatures

T	*Time to failure (TTF) of the system/component under consideration, a random variable,* $0 < E[T] < \infty$.
$f_T(t)$	*Probability density function (pdf) of the random variable* T
$F_T(t)$	*Cumulative Distribution Function (CDF) of the random variable* T*, can be identified as the unreliability* $U(t) = 1.0 - R(t)$
$R(t)$	*Reliability of the system/component under consideration;* $R(t) = P(T > t)$*;* $R(t) > 0$ *for* $t \geq 0$, $R(0) = 1.0$*;* $\lim_{t\to\infty} t\ R(t) = 0$*. Many characterizing function of* T *cab be obtained via algebraic manipulation, differentiation or integration of* $R(t)$
$r(t)$	*Failure rate (also called hazard rate* $h(t)$*),* $r(t) \geq 0$ *for* $t \geq 0$*. For many systems/components,* $r(t)$ *follows a bathtub curve, which starts by decreasing failure rate (DFR; infant mortality), followed by a constant failure rate (CFR; prime of life), and finally an increasing failure rate (IFR, wearout).*
$H(t)$	*Cumulative hazard rate (also called cumulative failure rate);* $H(t) = \int_0^t h(\tau)d\tau$
$m(t)$	*Mean residual life (MRL) (also called Remaining Useful Life (RUL), albeit this name might be undesirable since it has ambiguous economic connotations). The function* $m(t)$ *is also called residual mean time to failure (*$MTTF(t)$*), and is not to be confused with the mean time to failure (MTTF) which is a single number rather than a function of time.*
$G(t)$	*A useful characterizing function of the random variable* T*, defined as an integral function of reliability;* $G(t) = \int_t^\infty R(\tau)d\tau$.

1 Introduction

The survival time or time to failure (TTF), denoted by T, is an important non-negative continuous random variable such that $0 < E(t) < \infty$ that plays a central role in the overall area of general probability as well as its distinguished branch of reliability, which deals with the probability of systems and components performing their intended functions under specified conditions for a given period of time [1, 2]. The time to failure is also an indispensable concept in the emerging subfield of reliability engineering called (engineering) prognostics and health monitoring (PHM) (or, prognostics for short) [3], which deals mainly with the assessment of the remaining useful life (RUL) of components and systems. The term "prognostics" is borrowed from medicine, in which the term "(medical) prognostics" is of wide established utility [4, 5, 6, 7, 8]. Prognostic is generally defined as the process of predicting the future reliability of a product by assessing the extent of deviation or degradation of the product from its expected normal operating conditions.

It focuses on predicting the time in which the device will no longer perform its intended function. In the PHM Community, Prognostics is defined as the estimation of the Remaining Useful Life of a component.The Remaining Useful Life (RUL) is the amount of time a component can be expected to continue operating within its given specifications (Not necessarily a failure). Prognostics is used by industry to manage business risks that result from equipment failing unexpectedly.

This paper attempts to gather scattered information about the most important functions characterizing T. A tutorial exposition is given for the conceptual and mathematical definition of these functions as well as for the interrelations among them that allow each of them to be expressible in terms of each of the others. The paper is a useful introduction for any newcomers to the area of prognostics as well as a handy reference for well-established practitioners. It attempts to be self-contained in its compilation of various definitions and diverse and alternative formulas, and it includes many of the necessary fundamental derivations. The paper also tries to forward a unified set of notations, and to reconcile equivalent (albeit apparently dissimilar) concepts. We hope the paper might save prognostics students the trouble of handling intriguing questions about unclear relations concerning apparently diverse (albeit essentially equivalent) entities. The paper references constitute a small (hopefully representative) sample of the plethora of papers published recently in the topic.

The paper points out that there is currently a significant proliferation of characterizing functions emerged in various fields. It shows that, under mild conditions, the product and quotient of two characterizing functions are also characterizing functions. The choice of one characterizing function in a certain application in a matter of convenience and taste. Though we included many characterizing functions in this survey, we deliberately excluded many others. Notable among these excluded are the aging intensity function [9, 10], the log-odd rate [10] and entropy or entropy-related functions [11]. Our work herein hopefully supplements that in many useful expositions, surveys and reviews [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43].

The organization of the remainder of this paper is as follows. Section 2 defines and interrelates the most prominent functions characterizing T, including distribution functions (the Cumulative Distribution Function (CDF), and the probability density function (pdf)), the transform functions (the Moment Generating Function (MGF) and Characteristic Function), relaibaility, unreliability, and hazard (failure) rate. Section 3 compiles a subset of the set of characterizing functions that are frequently used in prognostics including the Mean Residual Life (MRL), typically referred to as the Remaining Useful Life (RUL) or as the residual Mean Time to Failure. Section 4 explains the phenomenon of proliferation of functions characterizing T. Section 5 concludes the paper.

2 Characterizing Functions

In this section, we briefly survey the most important functions used in characterizing the time to failure, denoted simply as T, as a general continuous random variable, as a variable pertaining to the general area of reliability engineering or to its subfield of prognostics engineering.

2.1 The distribution functions

The Cumulative Distribution Function (CDF) $F_T(t)$ is defined [2, 3, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] as:

$$F_T(t) = P(T \leq t), \qquad 0 \leq t < \infty \tag{2.1}$$

We note that the support of the nonnegative random variable T in Equation (2.1) is the semi-infinite interval $[0, \infty)$ and not the whole real line $\Re = (-\infty, \infty)$. The distribution $F_T(t)$ is a probability and hence it is dimensionless. It is a monotone non-decreasing (increasing) function of t that satisfies the properties

$$0 \leq F_T(t) \leq 1, \qquad 0 \leq t < \infty, \tag{2.2a}$$

$$F(0) = 0, \tag{2.2b}$$

$$\lim_{t \to \infty} F(t) = 1. \tag{2.2c}$$

The probability density function (pdf) $f_T(t)$ is defined by the limiting probability of failure in the interval $(t, t + \Delta t]$ divided by Δt, *i.e.*,

$$\begin{aligned} f_T(t) &= \lim_{\Delta t \to 0} \frac{P(t < T \leq t + \Delta t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{(P(T \leq t + \Delta t) - P(T \leq t))}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{F_T(t + \Delta t) - F_T(t)}{\Delta t} \\ &= dF_T(t)/dt. \end{aligned} \tag{2.3}$$

The pdf is not a probability but is a dimensional quantity of dimension:

$$[f_T(t)] = time^{-1}, \tag{2.4}$$

where we use the notation $[y]$ to denote the dimension of a quantity y. The pdf satisfies the properties

$$f_T(t) \geq 0, \qquad 0 \leq t < \infty, \tag{2.5a}$$

$$\int_0^\infty f_T(t) = 1. \tag{2.5b}$$

The inverse of the differentiation relation in Equation (2.3) is the integration relation

$$F_T(t) = \int_0^t f_T(\tau) d\tau. \tag{2.6}$$

In passing, we note that while $F_T(t)$ and $f_T(t)$ have no meaning for $t < 0$, they might be defined over the entire real line $(-\infty, \infty)$ while forcing them both to be identically 0 over $(-\infty, 0)$.

2.2 The transform functions

Transform characterization is particularly useful for computation of moments (without the need to evaluate tedious integrals) and for solving differential equations [22]. For the random variable T, the function $e^{T\theta}$ in another random variable, whose expectation, denoted $M(\theta)$ is

$$M(\theta) = E\left(e^{T\theta}\right) = \int_0^\infty e^{t\theta} f_T(t) dt, \tag{2.7}$$

is called the Moment Generating Function (MGF), and it is a dimensionless function that usually exits for at least some numbers θ. This function is so named since

$$M(\theta) = \sum_{k=0}^{\infty} E\left(T^k\right) \frac{\theta^k}{k!}, \tag{2.8}$$

under the assumption that all expectations in the RHS of Equation (2.8) exist. Equation (2.8) is a power-series expansion of $M(\theta)$ in terms of the k^{th} moment $E(T^k)$ of T $(k = 0, 1, 2, \dots)$ and might be rewritten as

$$E(T^k) = \frac{d^k M(\theta)}{d\theta^k}\bigg]_{\theta=0}, \qquad k = 0, 1, 2, \dots \tag{2.9}$$

Note that, in particular, we have

$$E(T^0) = \frac{d^0 M(\theta)}{d\theta^0}\bigg]_{\theta=0} = M(0) = E(e^0) = 1. \tag{2.10}$$

A function closely related to the MGF is the dimensionless characteristic function of the random variable T given by

$$N_T(w) = M_T(iw) = \int_0^\infty e^{iwt} f_T(t)dt, \tag{2.11}$$

where i stands for the imaginary unit $(i = \sqrt{-1})$, and w is a real variable. If we extend the definition of $f_T(\tau)$ over $(-\infty, 0)$ forcing it to be zero therein, then $N_T(w)$ is given by

$$N_T(w) = \int_{-\infty}^\infty e^{iwt} f_T(t)dt = \mathscr{F}(f_T(t)). \tag{2.12}$$

and the characteristic function is identified as the Fourier Transform of $f_T(t)$. Therefore, $f_T(t)$ is given as the inverse Fourier transform of $M_T(w)$, namely

$$f_T(t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-iwt} N_T(w)dw. \tag{2.13}$$

Another related characterization of the nonnegative continuous random variable T, is the dimensionless (one-sided) Laplace-Stieltjes transform of $f_T(t)$, given by

$$L_T(s) = M_T(-s) = \int_0^\infty e^{-st} f_T(t)dt = \mathscr{L}(f_T(t)). \tag{2.14}$$

where s is a complex variable $(s = \sigma + iw, \sigma > 0)$ called the complex frequency. The pdf $f_T(t)$ can inversely be obtained from $L_T(s)$ via an inverse formula requiring contour integration in the complex plane.

2.3 The reliability and unreliability functions

In reliability circles, the usual way to characterize T is to use the reliability function $R(t)$ (also called the survival function), which is the probability that the component or system survives until some time t [2, 3, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22], *i.e.*,

$$R(t) = P(T > t) = 1 - F_T(t) = \int_t^\infty f_T(t)dt. \tag{2.15}$$

Since the above definition implicitly assumes $R(0) = 1$, which means that the component or system is working or successful at time 0, (*i.e.*, it excludes the probability that the component or system is "*dead on arrival*" (see Equations (2.2b) and (2.15)), some authors argue that $R(t)$ should be interpreted as the conditional probability

$$R(t) = P(T > t | T > 0). \tag{2.16}$$

In fact, the concept of conditional probability is explicit when defining mission or conditional reliability [44], depicted $R(t_0 + t|t_0)$ or $MR(t_0, t)$, which is the probability that a system that already survived up to time t_0 will continue its survival for a further time of t, namely

$$\begin{aligned} R(t_0 + t|t_0) = MR(t_0, t) &= P(\text{success in}(0, t_0 + t]|\text{success in}(0, t_0]) \\ &= \frac{R(t_0 + t)}{R(t_0)} \end{aligned} \tag{2.17}$$

The definition above reduces to usual definition of reliability when $t_0 = 0$, *i.e.*,

$$R(t) = R(t|0) = MR(0, t) \tag{2.18}$$

The definition in Equation (2.15) demands the system to be successful in the semi-closed interval $(0, t]$. Since the underlying distribution is continuous $P(T = t) = 0$ {albeit the event $\{T = t\}$ is not an impossible event}, and hence the interval $(0, t]$ could be replaced by the doubly-open one $(0, t)$. If we view the condition $\{R(0) = 1\}$ as a requirement rather than an assumption, then the above interval could become $[0, t]$ or $[0, t)$. Therefore, four versions of the interval appear in the literature, which is somewhat confusing to scholars in the field. According to Equation (2.15), $R(t)$ is dimensionless, and is the Complementary Cumulative Distribution Function (CCDF), while its complement to one, called the unreliability $U(t)$, can be identified as $F_T(t)$. According to Equations (2.2a) and (2.15)), $R(t)$ is a monotonically non-increasing function of time, and according to Equations (2.2c) and (2.15)), the limiting value as $t \to \infty$ for $R(t)$ is

$$\lim_{t \to \infty} R(t) = 0, \tag{2.19}$$

i.e., no component or system can work forever without failure ($E[T] < \infty$). Usually, a stronger condition is imposed on $R(t)$, namely,

$$\lim_{t \to \infty} tR(t) = 0, \tag{2.20}$$

which means not only that $R(t)$ diminishes to 0 as t approaches ∞, but also that $R(t)$ does so faster than t^{-1}. The condition in Equation (2.20) is needed in the integration by parts of the formula for the mean time to failure:

$$\text{MTTF} = E[T] = \int_0^\infty t f_T(t) dt, \tag{2.21}$$

so as to arrive at the following celebrated expression for the MTTF.

$$\text{MTTF} = \int_0^\infty R(t) dt. \tag{2.22}$$

Strange conditions might also be imposed on $R(t)$ in order to facilitate the derivation of convenient expression for higher moments of T. Of course the MTTF is an aggregate metric that does not characterize T fully, but it is the most important single number (rather than function) in reliability studies. We will see shortly that it is generalized into a function that is of paramount importance in the study of prognostics.

2.4 The failure rate (The Hazard Rate)

The failure rate or hazard rate, denoted by $r(t)$ or $h(t)$ is defined such that $r(t)dt$ is the probability that an object that survives to age t fails in the interval $(t, t + dt]$, *i.e.*,

$$\begin{aligned} r(t) &= \lim_{\Delta t \to 0} P(t < T \leq t + \Delta t | T > t)/\Delta t \\ &= \lim_{\Delta t \to 0} \frac{P(t < T \leq t + \Delta t)}{\Delta t \; P(T > t)} \\ &= \lim_{\Delta t \to 0} \frac{P(T > t) - P(T > t + \Delta t)}{\Delta t \; R(t)} \\ &= \lim_{\Delta t \to 0} \frac{R(t) - R(t + \Delta t)}{\Delta t \; R(t)} \\ &= -\frac{dR(t)}{dt} \frac{1}{R(t)} \end{aligned} \tag{2.23}$$

Equation (2.23) can be rewritten as

$$r(t) = \frac{f_T(t)}{R(t)} \tag{2.24}$$

Equation (2.24) indicates a mathematical fundamental difference between $r(t)$ and $f_T(t)$. There is also a significant conceptual difference between these two quantities. The quantity $r(t)dt$ is the **conditional** probability that the component or system fails in the time interval $(t, t+dt]$ given that it has reached age t without failure. The quantity $f_T(t)dt$ is the **unconditional** probability that the component or system fails in the same time interval $(t, t + dt]$. Therefore, the pdf is sometimes called the mortality of the component or system. Obviously, $r(t) = f_T(t)$ at $t = 0$, but $r(t) > f_T(t)$ for $t > 0$, and $r(t) \gg f_T(t)$ as t increases and $R(t)$ approaches 0. Equation (2.23) can also be rewritten as a homogeneous first-order ordinary differential equation (ODE) in $R(t)$ with a variable coefficient $r(t)$

$$\frac{dR(t)}{dt} + r(t)\, R(t) = 0 \tag{2.25}$$

Together with the initial condition $R(0) = 1$, this ODE constitutes an initial value problem (IVP) whose solution is

$$R(t) = \exp\left(-\int_0^t r(\tau)d\tau\right) = \exp\left(-H(t)\right). \tag{2.26}$$

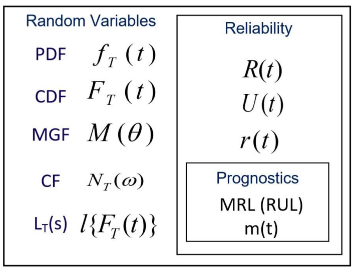

Fig. 1. A Venn diagram interrelating some Characterizing Functions for the time to failure

The plot of $r(t)$ versus age t is called the mortality (bathtub) curve (Fig. 2), and separates the time axis into three distinct periods described in Table 1. The useful constant failure rate (CFR) period is pronounced for electronic components but negligible (virtually non-existent) for mechanical and biological systems. Under the (extremely unrealistic) assumption of a CFR (exponential distribution) all throughout, the following results are obtained:

$$r(t) = \lambda = \text{constant}, \qquad t \geq 0 \tag{2.27}$$

$$R(t) = e^{-\lambda t}, \qquad t \geq 0 \tag{2.28}$$

$$\text{MTTF} = \frac{1}{\lambda}, \tag{2.29}$$

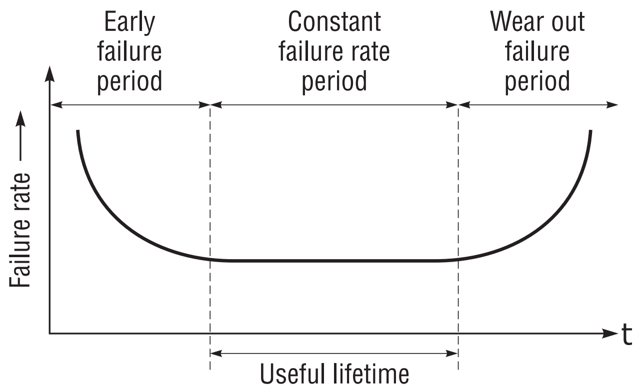

Fig. 2. The mortality (bathtub) curve for a component or system (see e.g., [24]).

The failure rate $r(t)$ should be modeled by more realistic distributions such as the Weibull [23]. Many references are dedicated to the issue of selecting appropriate distributions to model the entire bathtub curve or certain portions thereof (see e.g., [24]).

Table 1. Periods of operating life

Description \ Period	**Early Life**	**Useful Life**	**Wearout**
Alternative Name	Infant mortality Burn-in period Break-in period Shake-down period	Prime-of-life	Degradation period Deterioration period
Slope of the mortality curve	Negative DFR: Decreasing Failure Rate	Zero CFR: Constant Failure Rate	Positive IFR: Increasing Failure Rate
Nature of failures	Due to design or manufacturing weaknesses or faults	Random, Catastrophic, or unpredictable	Due to aging or wear

2.5 The mean residual life

The mean residual life (MRL) $m(t)$ is the main characterizing function used in prognostics, and it appears, in disguise under alternative names, in many related disciplines [17]. It is sometimes known under the alternative name of the Remaining Useful Life (RUL). However, the term MRL is a more neutral term that is not lacking mathematical preciseness but lacking economic pertinence instead. Generally speaking, the RUL is the period of time, from the current time to the time of termination of the component or system (including termination due to inadequate performance) expressed as a ratio of its expected useful life (depending on the context) and operational characteristics) [17]. The MRL $m(t)$ has the (almost equivalent) mathematical definition (for $E[T] < \infty$):

$$\begin{aligned} m(t) &= E[T - t|T > t] \\ &= \frac{\int_t^\infty (\tau - t) f_T(\tau) d\tau}{P(T > t)} \\ &= \frac{\int_t^\infty R(\tau) d\tau}{R(t)} \end{aligned} \tag{2.30}$$

Since division by 0 is not admissible, some authors use Equation (2.30) provided that $R(t) \neq 0$ and set $m(t) = 0$ when $R(t) = 0$. The final result in Equation (2.30) is obtained via integration by parts and demanding that Equation (2.20) be satisfied. An alternative version of Equation (2.30) in terms of the pdf $f_T(t)$ is simply obtained via rearrangement without integration as

$$m(t) = \frac{\int_t^\infty \tau f_T(\tau) d\tau}{R(t)} - t. \tag{2.31}$$

Some authors define the MRL as the residual MTTF, *i.e.*, the expected time to failure of a component or system aged t, depicted $MTTF(t)$, namely

$$MTTF(t) = m(t) = \int_0^\infty R(t + \tau|t) d\tau = \int_t^\infty (\tau - t) f_T(\tau|t) d\tau \tag{2.32}$$

where $f_T(\tau|t) = f_T(\tau)/R(t)$ is the pdf of conditional probability of failure at time τ provided that the component or system survived up to time t. The definition in Equation (2.32) is obviously in agreement with the definition in Equation (2.30). The numerator in Equation (2.30) given by

$$G(t) = \int_t^\infty R(\tau) d\tau = \int_0^\infty R(\tau - t) d\tau = m(t)R(t), \tag{2.33}$$

which is another characterization function, whose time derivative is given by

$$\frac{dG(t)}{dt} = -R(t) = -\frac{1}{m(t)} G(t), \tag{2.34}$$

and hence it satisfies the homogeneous first-order ODE

$$\frac{dG(t)}{dt} + \frac{1}{m(t)} G(t) = 0, \tag{2.35}$$

which together with the initial condition $G(0) = m(0)R(0) = m(0)$ constitutes an initial value problem IVP with the solution

$$G(t) = m(0) \exp\left(-\int_0^t \frac{d\tau}{m(\tau)} \right), \tag{2.36}$$

which allows us to produce the following Inverse Formula for the reliability (survival) function

$$R(t) = \frac{G(t)}{m(t)} = \frac{m(0)}{m(t)} \exp\left(-\int_0^t \frac{d\tau}{m(\tau)}\right). \tag{2.37}$$

Another way to derive the inversion Formula (2.37) is to write

$$\begin{aligned} -\int_0^t \frac{d\tau}{m(\tau)} &= -\int_0^t \frac{R(\tau)}{G(\tau)} d\tau = \int_0^t \frac{(dG(\tau)/d\tau)}{G(\tau)} d\tau \\ &= [\ln(G(\tau))]_0^t = \ln\left(\frac{G(t)}{G(0)}\right) \\ &= \ln\left(\frac{R(t)m(t)}{R(0)m(0)}\right) = \ln\left(\frac{R(t)m(t)}{m(0)}\right) \end{aligned} \tag{2.38}$$

Exponentiation of both sides of this result produces

$$\frac{R(t)m(t)}{m(0)} = \exp\left(-\int_0^t \frac{d\tau}{m(\tau)}\right), \tag{2.39}$$

which can be rearranged to produce Equation (2.37).

Another Useful relation is obtained when $m(t)$ is differentiable, since

$$\begin{aligned} \frac{dm(t)}{dt} &= \frac{d}{dt}\left(\frac{G(t)}{R(t)}\right) = \frac{\frac{dG(t)}{dt} R(t) - G(t)\frac{dR(t)}{dt}}{R^2(t)} \\ &= \frac{-R^2(t) + (m(t)R(t))\,(r(t)R(t))}{R^2(t)} \\ &= -1 + m(t)r(t), \end{aligned} \tag{2.40}$$

and hence $r(t)$ is expressed in terms of $m(t)$ as

$$r(t) = \frac{\frac{dm(t)}{dt} + 1}{m(t)}. \tag{2.41}$$

Equation (2.40) can also be obtained by differentiating both sides of Equation (2.39). Alternatively, Equation (2.40) can be viewed as an inhomogeneous first-order ODE whose solution under appropriate initial conditions is given by Equation (2.39).

3 Characterizing Functions in Prognostics

There is a plethora of characterizing functions of T that can be used in prognostics. For easy reference, we compile in Table 2 interrelations among six such quantities, and then survey in Table 3 certain properties of the four quantities among them that are most prominent in prognostics. The properties covered include initial values, dimensions, algebraic relations, derivative relations and integral relations.

Table 2. Interrelations among six characterizing functions

From \ To	$\boldsymbol{f_T(t)}$	$\boldsymbol{F_T(t) \equiv U(t)}$	$\boldsymbol{R(t)}$	$\boldsymbol{r(t)}$	$\boldsymbol{G(t)}$	$\boldsymbol{m(t)}$
$\boldsymbol{f_T(t)}$	$f_T(t)$	$\int_0^t f_T(\tau)d\tau$	$\int_t^\infty f_T(\tau)d\tau$	$\frac{f_T(t)}{\int_t^\infty f_T(\tau)d\tau}$	$\int_t^\infty \left(\int_\tau^\infty f_T(x)dx\right) d\tau$	$\frac{\int_t^\infty \left(\int_\tau^\infty f_T(x)dx\right)d\tau}{\int_t^\infty f_T(\tau)d\tau}$
$\boldsymbol{F_T(t)}$	$\frac{dF_T(t)}{dt}$	$F_T(t)$	$1 - F_T(t)$	$\frac{\frac{dF_T(t)}{dt}}{1-F_T(t)}$	$\int_t^\infty (1 - F_T(\tau))d\tau$	$\frac{1}{1-F_T(t)} \int_t^\infty (1 - F_T(\tau)d\tau)$
$\boldsymbol{R(t)}$	$-\frac{dR(t)}{dt}$	$1 - R(t)$	$R(t)$	$-\frac{1}{R(t)} \frac{dR(t)}{dt}$	$\int_t^\infty R(\tau)d\tau$	$\frac{1}{R(t)} \int_t^\infty R(\tau)d\tau$
$\boldsymbol{r(t)}$	$r(t) \exp\left[-\int_0^t r(\tau)d\tau\right]$	$1 - \exp\left[-\int_0^t r(\tau)d\tau\right]$	$\exp\left[-\int_0^t r(\tau)d\tau\right]$	$r(t)$	$\int_t^\infty \exp\left[-\int_0^\tau r(x)dx\right]d\tau$	$\frac{\int_t^\infty \exp\left[-\int_0^\tau r(x)dx\right]d\tau}{\exp\left[-\int_0^t r(\tau)d\tau\right]}$
$\boldsymbol{G(t)}$	$\frac{d^2G}{dt^2}$	$1 + \frac{dG(t)}{dt}$	$-\frac{dG(t)}{dt}$	$-\frac{\frac{d^2G}{dt^2}}{\frac{dG(t)}{dt}}$	$G(t)$	$-\frac{G(t)}{\frac{dG(t)}{dt}}$
$\boldsymbol{m(t)}$	$\frac{m(t)\left[\frac{dm(t)}{dt}+1\right]}{m^2(t)} \exp - \int_o^t \frac{d\tau}{m(\tau)}$	$1 - \frac{m(0)}{m(t)} \exp\left[-\int_0^t \frac{d\tau}{m(\tau)}\right]$	$\frac{m(0)}{m(t)} \exp\left[-\int_0^t \frac{d\tau}{m(\tau)}\right]$	$\frac{\left(\frac{dm(t)}{dt}-1\right)}{m(t)}$	$m(0) \exp\left[-\int_0^t \frac{d\tau}{m(\tau)}\right]$	$m(t)$

Table 3. Properties of the four most prominent characterizing function in prognostics

Quantity	Initial Value	Dimension	Algebraic Expression	Derivative Relations	Integral Relations
Reliability $R(t)$	$R(0) = 1$	Dimensionless ($[Time]^0$)	$F_T(t) = 1 - R(t)$	$f_T(t) - \frac{dR(t)}{dt}$ $r(t) = \frac{-1}{R(t)} \frac{dR(t)}{dt}$	$E[t] = MTTF = \int_0^\infty R(t)dt$ $G(t) = \int_t^\infty R(\tau)d\tau$
Failure Rate $r(t)$	$r(0) = f_T(0)$	$[Time]^{-1}$	$R(t) = \frac{f_T(t)}{r(t)}$		$CFR = \int_0^t r(\tau)d\tau$ $CFR = \ln\left(\frac{1}{R(t)}\right)$
Mean Residual Life (MRL) $m(t)$	$m(0) = E[T]$	$[Time]^1$	$m(t) = \frac{G(t)}{R(t)}$	$R(t) = -\frac{dG(t)}{dt}$	
Product of Reliability & MRL $G(t)$	$G(0) = E[T]$	$[Time]^1$	$G(t) = R(t)m(t)$	$m(t)r(t) = 1 + \frac{dm(t)}{dt}$	$R(t) = \frac{m(0)}{m(t)} e^{\left[-\int_0^t \frac{1}{m(\tau)} d\tau\right]}$

4 On the Proliferation of Characterizing Functions

The fact that $C_1(t)$ and $C_2(t)$ are two characterizing functions of the random variable T, means that each of them can expressed as a function of the other, *i.e.*,

$$C_1(t) = f\left(C_2(t)\right) \tag{4.1}$$

$$C_2(t) = f^{-1}\left(C_1(t)\right) \tag{4.2}$$

Now many functions can be generated from $C_1(t)$ and $C_2(t)$ such as to be also characterizing functions of T. Notable (and simple) among these are the product $p(t)$ and quotient $q(t)$ of $C_1(t)$ and $C_2(t)$, *i.e.*,

$$p(t) = C_1(t) \times C_2(t), \tag{4.3}$$

$$q(t) = C_1(t)/C_2(t), \tag{4.4}$$

which can be written as

$$\begin{aligned} p(t) &= C_1(t) \times f^{-1}\left(C_1(t)\right) \\ &= F_p\left(C_1(t)\right), \end{aligned} \tag{4.5}$$

$$\begin{aligned} q(t) &= C_1(t)/f^{-1}\left(C_1(t)\right) \\ &= F_q\left(C_1(t)\right). \end{aligned} \tag{4.6}$$

If the functions F_p and F_q are invertible, then $C_1(t)$ can be expressed in terms of either $p(t)$ or $q(t)$, *i.e.*,

$$C_1(t) = F_p^{-1}\left(p(t)\right), \tag{4.7}$$
$$C_1(t) = F_q^{-1}\left(q(t)\right), \tag{4.8}$$
$$\tag{4.9}$$

A particularly notable example of this possibility is when $C_1(t)$ is the failure rate $r(t)$, and $C_2(t)$ is the MRL $m(t)$, for then their product $r(t)m(t)$ and their quotient $r(t)/m(t)$ are each a characterizing function of T. More accessible examples (that we have seen earlier in Section 3) is when $C_1(t) = m(t)$, $C_2(t) = R(t)$, and $p(t) = m(t)R(t) = G(t)$, or when $C_1(t) = r(t)$, $G_2(t) = R(t)$, and $p(t) = r(t)R(t) = f_T(t)$.

For convenience, Fig. 1 shows some of the well known characterizing functions for T surveyed herein within the overall area of general probability, the restricted area of reliability, and finally the subfield of engineering prognostics.

5 Conclusions

The time to failure of a specific component/system can be fully described as any random variable by either the probability density function (pdf), the Cumulative Distribution Function (CDF), the moment generating function, or the characteristic function. However, due to the special nature of this variable, it is more conveniently described *via* the reliability function, which is its Complementary Cumulative Distribution Function (CCDF). In certain occasions, it is even more convenient to use other functions derived from the reliability function *via* differentiation (the failure or hazard rate function) or integration (the mean residual life function). The aim of this paper is to interrelate and compare the plethora of functions frequently used in reliability, in general, and in prognostics, in particular. These functions are interrelated mathematically such that each of them is expressible in terms of any of the others. Each of these functions has perhaps its own merits and advantages that make it suitable (or preferable) in particular applications or subfields. Such suitability (or preferability) is usually something in the eyes of the beholder, *i.e.*, it is often a mater of taste, convenience, or discretion of the user.

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Competing Interests

Authors have declared that no competing interests exist.

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