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Geometric Aspects of Denseness Theorems for Dirichlet Functions

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Authors' contributions

This work was carried out in collaboration between both authors. Author DG designed the study and managed the literature searches. Author AHM performed managed the analyses of the study, wrote the protocol, and wrote the first draft of the manuscript. Both authors read and approved the final manuscript.

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Opinion Article

Abstract

The first theorem related to the denseness of the image of a vertical line $\operatorname{Re} s = \sigma_0$, $\sigma_0 > 1$ by the Riemann Zeta function has been proved by Harald Bohr in 1911. We argue that this theorem is not really a denseness theorem. Later Bohr and Courant proved similar theorems for the case $1/2 < \operatorname{Re} s \leq 1$. Their results have been generalized to classes of Dirichlet functions and are at the origin of a burgeoning field in analytic number theory, namely the universality theory. The tools used in this theory are mainly of an arithmetic nature and do not allow a visualization of the phenomena involved. Our method is based on conformal mapping theory and is supported by computer generated illustrations. We generalize and refine Bohr and Courant results.

Keywords: Denseness theorems; Diophantine approximation; general Dirichlet series; fundamental domains.

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1 Introduction

Bohr theorem, see [1], says that for any given complex number $z \in \mathbb{C}\backslash\{0\}$ there is $\sigma_0 > 1$ such that for any $\varepsilon > 0$, a $t_0 \in \mathbb{R}$ can be found for which $|\zeta(\sigma_0 + it_0) - z| < \varepsilon$. In other words the image $\zeta(\sigma_0 + it)$ of the line $\operatorname{Re} s = \sigma_0$ gets any closer to z for conveniently chosen $t \in \mathbb{R}$. We notice that σ_0 depends on z and once fixed, there is no reason to expect that the same line will get closer to some other arbitrary complex number. Therefore, it cannot be inferred that there is any $\sigma_0 > 1$ for which the set $\{\zeta\,(\sigma_0 + it)\,|\,t \in \mathbb{R}\}$ is dense in some domain of the complex plane. However, the computer experimentation (see Fig. 1 and Fig. 2) suggests that a bounded domain can exist in which the image of $\operatorname{Re} s = \sigma_0$ is a dense set. Yet Bohr theory does not deal with such a problem.

Another theorem proved by Bohr in [1] says that for every $z \in \mathbb{C}\backslash\{0\}$ and every $\delta > 0$ we have $\zeta(s) = z$ for some s with $1 < \operatorname{Re} s < 1 + \delta$, i.e. not only the image of this strip is dense in $\mathbb{C}$, but it effectively covers the whole dotted plane $\mathbb{C}\backslash\{0\}$. Later Bohr and Courant, see [2], proved similar theorems for the case $1/2 < \operatorname{Re} s \leq 1$. Their results have been generalized to classes of Dirichlet functions and are at the origin of a burgeoning field in analytic number theory, namely the universality theory.

Bohr and Courant theorems do not say that for the given z and the corresponding σ_0 there is t_0 such that $\zeta(\sigma_0 + it_0)$ is effectively equal to z. Yet, taking $\varepsilon_n \to 0$, they insure that a sequence (t_n) exists such that $|\zeta(\sigma_0 + it_n) - z| < \varepsilon_n$, therefore $\lim_{n\to\infty} \zeta(\sigma_0 + it_n) = z$. It may happen that there is a convergent subsequence (t_{n_k}) of (t_n) and if $\lim_{k\to\infty} t_{n_k} = t_0$ then, by the continuity of the function $\zeta(s)$, we have that $\zeta(\sigma_0 + it_0) = z$. If such a subsequence does not exist, then a sequence (t_n) must have the limit $+\infty$ or $-\infty$ and $\lim_{n\to\infty} \zeta\,(\sigma_0 + it_n) = z$. One can say that the value z is always reached by $\zeta(\sigma + it)$ on some line $\operatorname{Re} s = \sigma_0$, $\sigma_0 > 1$, either in a finite interval for t, or at the limit as $t \to \infty$. The question remains: does the image of a line $\operatorname{Re} s = \sigma_0$ reach any complex value? In the affirmative case we will say that *the line has the denseness property in the whole complex plane.*

The tool used by Bohr and Courant to prove their denseness theorems has been Kronnecker's Diophantine approximation which states that given a set of linearly independent (in the field of rationals) real numbers $\mu_1, \mu_2, ..., \mu_n$ and a set of arbitrary real numbers $\eta_1, \eta_2, ..., \eta_n$, for every $\varepsilon > 0$ there is $t_0 \in \mathbb{R}$ and a set of integers $g_1, g_2, ..., g_n$ such that $|t_0\mu_k - \eta_k - g_k| < \varepsilon$ for every $k = 1, 2, ..., n$. This means that $t_0\mu_k = \eta_k + g_k + \alpha_k$, where $|\alpha_k| < \varepsilon$ and since $e^{2g_k\pi i} = 1$, we have $e^{2\pi t_0\mu_k i} = e^{2\pi\eta_k i} \cdot e^{2\pi\alpha_k i}$, where $|e^{2\pi\alpha_k i}|$ is close to 1. Bohr started in [1] with the construction of an infinite product depending on σ_0, on the sequence (p_n) of prime numbers and on some real numbers η_n ingeniously chosen such that the limit of the product is the given complex number z

$$F(\eta_1, \eta_2, ..., \eta_n, ...) = \prod_{n=1}^{\infty} (1 + p_n^{-\sigma_0} e^{2\pi i\eta_n})^{-1} = z. \tag{1.1}$$

Then, for an arbitrary $\varepsilon > 0$ a natural number N can be found such that

$$\prod_{n=N+1}^{\infty} (1 + p_n^{-\sigma_0} e^{2\pi i\eta_n})^{-1} < \frac{\varepsilon}{4 + \varepsilon} = \varepsilon_1. \tag{1.2}$$

In order to compare the product $\prod_{n=1}^{N} (1 + p_n^{-\sigma_0} e^{2\pi i\eta_n})^{-1}$ with $\zeta(\sigma_0 + it)$, he used Diophantine approximation for the the first N terms of the Euler product expressing this function:

$$\zeta(\sigma_0 + it) = \prod_{n=1}^{\infty} (1 + p_n^{-\sigma_0} e^{i\mu_n})^{-1},$$

where $\mu_n = \pi - t \log p_n$.

On the page 418 he writes:
Ferner ist für jedes reelle t

$$\prod_{n=N+1}^{\infty}(1+p_n^{-\sigma_0}e^{i\mu_n})=1+\alpha_2, \tag{1.3}$$

wo $\alpha_2=\alpha_2(t)$ *die Ungleichung* $|\alpha_2|<\varepsilon_1$ *erfüllt.*

We notice that this affirmation can be true only if $\prod_{n=1}^{\infty}(1+p_n^{-\sigma_0}e^{i\mu_n})$ converges uniformly with respect to $t\in\mathbb{R}$, which is the case. Indeed, the inequality

$$\left|\sum_{n=1}^{\infty}\frac{1}{n^{\sigma_0+it}}\right|\leq\sum_{n=1}^{\infty}\frac{1}{n^{\sigma_0}}=\zeta(\sigma_0)$$

implies the uniform convergence of the series $\sum_{n=1}^{\infty}\frac{1}{n^{\sigma_0+it}}$ for $t\in\mathbb{R}$, which in turn implies the uniform convergence of the Euler product $\prod_{n=1}^{\infty}(1+p_n^{-\sigma_0}e^{i\mu_n})^{-1}$ representing the same function. This last inequality shows that the image of the line $\operatorname{Re}s=\sigma_0$ by the function $\zeta(s)$ is a bounded set, and therefore it cannot be dense in the whole complex plane.

Thus, the interpretation of the sentence:

zu jedem $z\neq 0$ *gibt es eine reelle Zahl* $\sigma_0>1$ *derart, dass auf der Geraden* $\sigma=\sigma_0$ *die Function* $\zeta(s)-z$ *beliebig kleine Werte annimmt*

should not be in terms of denseness. To insure this and taking into account the previous remark, we can state the following:

Theorem 1.1. *For no value* $\sigma_0>1$*, has the line* $\operatorname{Re}s=\sigma_0$ *the denseness property in the whole complex plane.*

Moreover, even for the case $\frac{1}{2}<\sigma_0\leq 1$, which will be dealt with in section 2, the existence of t_0 with the given property must be treated with care. The value t_0 depends on η_k, which depend on σ_0 and the number N depends on ε. But t_0 depends also on μ_k, which in turn depend on t. This chain of dependence is extremely intricate.

We will show that, for example, if z is real negative the existence of t_0 is rather improbable. Indeed, the existence of t_0 implies that the image of the line $\operatorname{Re}s=\sigma_0$ passes through that z, i.e. the line $\operatorname{Re}s=\sigma_0$ intersects the pre-image of the negative real half axis at a point corresponding to a negative value as big as we want in absolute terms. Yet, if instead of $\operatorname{Re}s=\sigma_0$ we take a half-line making an arbitrary small angle with the vertical, since $\lim_{\sigma\to+\infty}\zeta(\sigma+it)=1$ uniformly with respect to t, the image of an infinite part of that half line will remain in a neighborhood of $z=1$ for t big enough. Hence only a finite number of points on this half-line can be on the pre-image of the negative half axis. To justify this affirmation we need to make use of the geometric characterization of the mapping of the complex plane by the function $\zeta(s)$ (see [3], [4], [5] and Figs 1 and 2). It is known that the mapping is locally injective, except for the points where $\zeta'(s)=0$, which forms a discrete set. This set and the pre-image of the real axis provide a partition of the complex plane into sets whose interior are mapped conformally by $\zeta(s)$ onto the whole complex plane with some slits. Such domains have been found for any Dirichlet series having a half plane of convergence and which can be continued analytically to the whole complex plane except for a simple pole at $s=1$. These are the fundamental domains of the function. Every fundamental domain contains a unique component of the pre-image of the negative real half axis. Such a component may or may not intersect the line $\operatorname{Re}s=\sigma_0$, where $\sigma_0>\frac{1}{2}$.

The computer experimentations with the Riemann Zeta function we conducted so far shows very rare events of such intersections. In fact, in our trials, no such intersection occurred for $\sigma_0 > 1$. Moreover, as seen in Figs. 1 and 2 below, where we took $\sigma_0 = 1.01$, there is an insignificant difference between the images by $\zeta(s)$ of the intervals $[10^6, 10^6 + 100]$ and $[10^9, 10^9 + 100]$ on the ordinate in regard with the interval spanned by the respective images on the real axis. Moreover, these last intervals are both on the positive real half axis. There is no obvious reason why the configuration should change drastically for even bigger values of t such that the image by $\zeta(s)$ of the line $\mathrm{Re}\, s = 1.01$ should hit the negative real half axis.

Similar images of very distant intervals on $\mathrm{Re}\, s = 1.01$

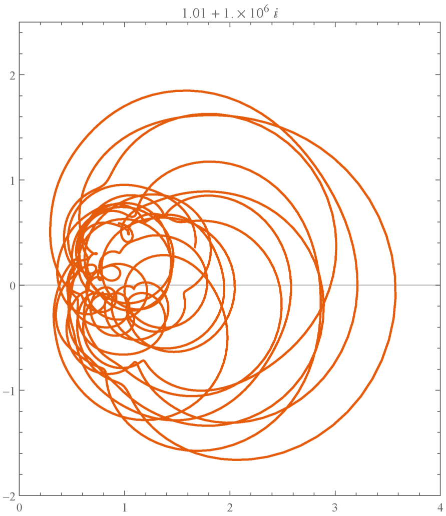


Fig. 1. Plot of Riemann Zeta Function for $\sigma_0 = 1.01$ and $t = \mathrm{Im}\,(s) \in \left[10^6, 10^6 + 100\right]$

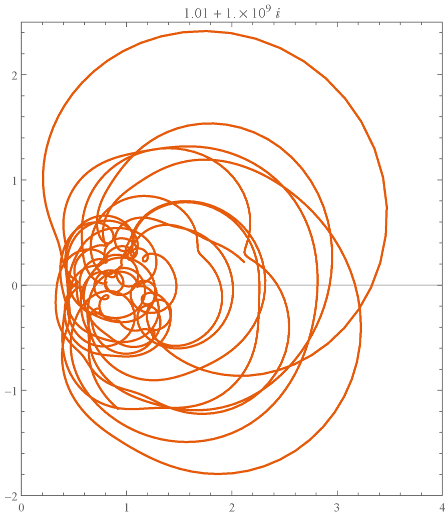


Fig. 2. Plot of Riemann Zeta Function for $\sigma_0 = 1.01$ and $t = \mathrm{Im}\,(s) \in \left[10^9, 10^9 + 100\right]$

2 The Case of the Critical Strip

As the image by $\zeta(s)$ of a line $\mathrm{Re}\, s = \sigma_0 + it$ with σ_0 in the critical strip is no longer bounded, it can be expected that the respective line has the denseness property. But the method used by Bohr in his first paper on this topic was no more applicable in such a case since the Riemann series is divergent for $\sigma \leq 1$. He and Courant succeeded to circumvent this difficulty by using a mean value theorem which says that if the Dirichlet series $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges for $\mathrm{Re}\, s > 0$, then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^2 \, dt = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}}.$$

They found that, when

$$f(s) = \left\{ \zeta(s) \prod_{n=1}^{N} (1 - p_n^{-s}) - 1 \right\} (1 - 2^{1-s}),$$

the coefficients a_n are such that $a_n = 0$ for $n < p_{N+1}$ and $|a_n| \leq 2$ for $n \geq p_{N+1}$, which allowed them an easy evaluation of that mean value and finally to deal with the case where $\frac{1}{2} < \sigma_0 \leq 1$.

Bohr and Courant denseness theorems have an interesting implication regarding the fundamental domains of the function $\zeta(s)$. The computer experimentation in [3], [4], [5] has shown that all the S_k-strips of this function have a width on the line $\mathrm{Re}\, s = 1$ of approximately 10 and that the number of fundamental domains contained in those strips is increasing apparently logarithmically with k. There is no proof for these findings, but if they are true, then the width of some fundamental domains on the line $\mathrm{Re}\, s = 1$ must tend to zero as $k \to \infty$.

We can show now that this is true regardless of the previously mentioned experimental observations. We use as tool the invariance of conformal module of quadrilaterals with respect to a conformal mapping (see [6], page 19). Since $\{\zeta(1+it) | \, t \in \mathbb{R}\}$ is an unbounded set and for any $\delta > 0$, the set $\{\zeta\,(1+\delta+it) \,|\, t \in \mathbb{R}\}$ is bounded, a sequence of quadrilaterals whose conformal module form an unbounded set can be defined. Indeed, there is a sequence (t_n) such that $\lim\limits_{n\to\infty} \zeta(1+it_n) = \infty$. Let us denote by Ω_n the fundamental domains whose closure contains $1+it_n$. Obviously, we can change t_n such that for $\delta > 0$ small enough, both $1+it_n$ and $1+\delta+it_n$ belong to Ω_n. Let Δ_n be the quadrilateral having two vertices at $1+it_n$ and $1+\delta+it_n$, the other two at the intersection of the lines $\mathrm{Re}\, s = 1$ and $\mathrm{Re}\, s = 1+\delta$ with one of the components of the boundary of Ω_n and the sides on the respective component, on the lines $\mathrm{Re}\, s = 1$, $\mathrm{Re}\, s = 1+\delta$ and on the line $\mathrm{Im}\, s = t_n$. Since the image by $\zeta(s)$ of the line $\mathrm{Re}\, s = 1+\delta$ is bounded one of the sides of the quadrilaterals $\zeta(\Delta_n)$ remains bounded as $n \to \infty$ and since $\lim\limits_{n\to\infty} \zeta(1+it_n) = \infty$, the other three sides have the lengths tending to ∞ as $n \to \infty$. It results that the conformal module of $\zeta(\Delta_n)$ tends to ∞ as $n \to \infty$. The same must happen with the conformal module of Δ_n. Since one of the sides has the fixed length δ, the length of the adjacent sides must tend to zero and this proves our affirmation.

We need now to take a closer look at the denseness property in the critical strip.

When $\sigma_0 = 0.51$ only small in absolute terms negative values for the intersection of the image by $\zeta(s)$ of the line $\mathrm{Re}\, s = 0.51$ with the real axis were involved when letting t vary through values in the range of 10^{12} (see Fig. 4).

Moreover, in the S_k-strip shown in Fig. 3, which contains 22 non trivial zeros of $\zeta(s)$, the line $\mathrm{Re}\, s = 0.51$ does not intersect any component of the pre-image of the negative real half axis, therefore its image by $\zeta(s)$ does not intersect the negative real half axis. Only two instances appear in this strip where a line $\mathrm{Re}\, s = \sigma_0$, $\sigma_0 > 0.5$ can hit a component of the pre-image of the negative real half axis.

The arithmetic approach, by using Kronnecker theorem, does not allow any geometric interpretation. However, the phenomenon is of a geometric nature, since it involves the conformal mapping, hence we should be able to express geometrically the denseness property. One way to do it is to look at the image by $\zeta(s)$ of segments of the line $\mathrm{Re}\, s = \sigma_0$ included in every strip S_k (see [3], [4], [5]).

It is a curve starting on the interval $(1, +\infty)$ of the real axis, ending on the same interval, having several other points of intersection with the real axis and several self-intersection points. Illustrations of such curves appear in Figs. 1,2, and 4,5. The image of every segment of that line included in a fundamental domain is a Jordan arc, since the mapping is injective in the respective domain. These are parts of that curve having no self-intersection points, although different such Jordan arcs can intersect each other. Also, as seen in Fig. 4, some of these arcs intersect the negative real half axis and then, since every component of the pre-image of the negative real half axis belongs to a unique fundamental domain, they must intersect it twice or be tangent to it. Since $\zeta(s)$ is an analytic function, the image by $\zeta(s)$ of any line $\operatorname{Re} s = \sigma_0$ is smooth, except at the images by $\zeta(s)$ of the zeros of $\zeta'(s)$. In such an image the curve must have a turning point with a unique half-tangent, as seen in Fig. 6 and 7. If the line $\operatorname{Re} s = \sigma_0$ passes close to a zero of $\zeta'(s)$, then its image has a *false* turning point, as those which can be noticed in Fig. 1 and Fig. 2 or the curve can make a small loop there. In fact the curve continues to be smooth in the neighborhood of such a zero. For $\sigma_0 > \frac{1}{2}$, if the line $\operatorname{Re} s = \sigma_0$ intersects twice $\Gamma_{k,j}$, then both points of intersection are on the same component of the pre-image, either of the negative real half axis or of the positive real half axis. For no apparent reason, the last one is much more frequent. In fact, the computer experimentation did not show at all instances of the first one for values of the ordinate t up to 10^6. However, for bigger values of t, as for example 10^{12} in Fig. 3, their denseness in a small interval at the left of $z = 0$ is conceivable. Yet, for a larger interval the values of t have to be exceptionally big.

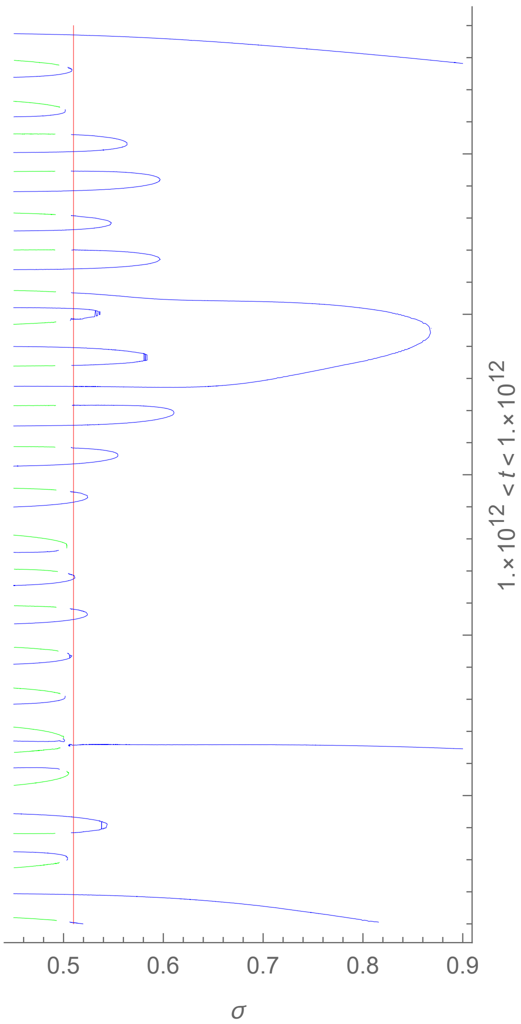


Fig. 3. A srtip S_k with 22 zeros

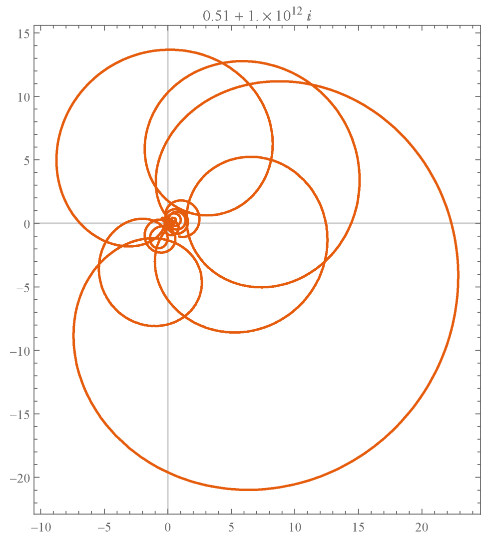


Fig. 4. Plot of Riemann Zeta Function for $\sigma_0 = 0.51$ and $t = \mathrm{Im}\,(s) \in \left[10^{12} + 5, 10^{12} + 10\right]$

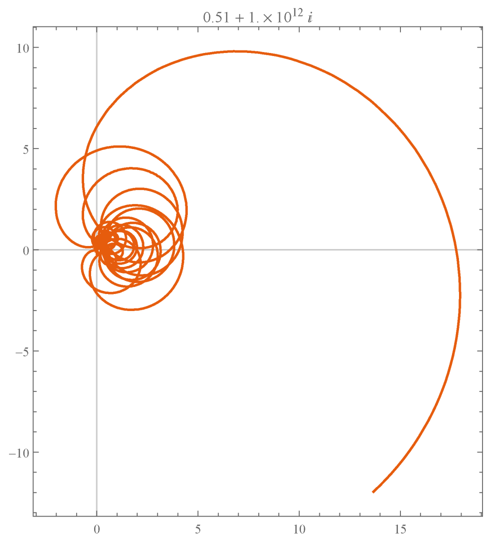


Fig. 5. Plot of Riemann Zeta Function for $\sigma_0 = 0.51$ and $t = \mathrm{Im}\,(s) \in \left[10^{12} + 11.2, 10^{12} + 16.8\right]$

3 General Dirichlet Series

We have dealt in [4] and [8] with different geometric aspects of general Dirichlet series and have shown that most of the properties of the Riemann Zeta function extend to vast classes of such series. Moreover, the Dirichlet L-functions have been implemented in *Mathematica* and the possibility exists of producing computer visualizations of most phenomena similar to those known for the

Image of a vertical line passing through double zero $s_{\circledcirc} = 0.5 + 31.586925 \cdot i$.

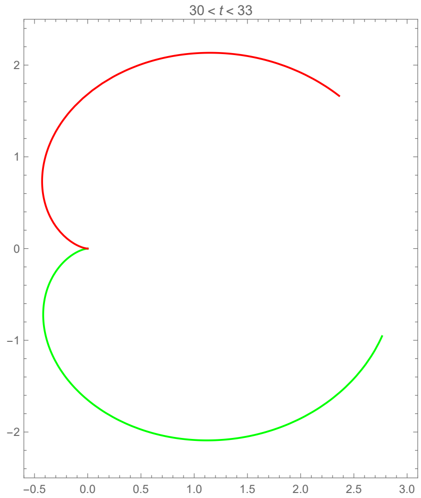


Fig. 6. Parametric plot of f on $\left\{\frac{1}{2} + i \cdot t \middle| t \in [30, 33]\right\}$

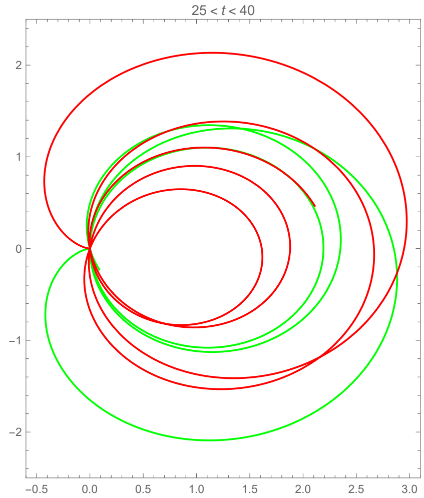


Fig. 7. Parametric plot of f on $\left\{\frac{1}{2} + i \cdot t \middle| t \in [25, 40]\right\}$

function $\zeta(s)$. In particular, we can illustrate the denseness property of vertical lines for these functions. Due to the fact that to all the functions obtained by analytic continuation of general Dirichlet series fundamental domains which are horizontal strips are associated, the image of a vertical line will be always similar to those in Figs. 1, 2 and 4, 5. Indeed, it is the union of infinitely many Jordan arcs, which correspond to the part of that line contained in a fundamental domain. The denseness property means simply that these Jordan arcs fill the whole complex plane.

A general Dirichlet series is an expression of the form

$$\zeta_{A,\Lambda}(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \tag{3.1}$$

where $A = (a_n)$ is an arbitrary sequence of complex numbers with $a_1 = 1$ and $\Lambda = (1 = \lambda_1 < \lambda_2 < \ldots)$ is an increasing sequence of positive numbers such that $\lim_{n\to\infty} \lambda_n = \infty$. Suppose that the abscissa of absolute convergence of $\zeta_{A,\Lambda}(s)$

$$\sigma_a = \limsup_{n\to\infty} \frac{1}{\lambda_n} \log \sum_{k=1}^{n} |a_k| \tag{3.2}$$

is finite and that $\zeta_{A,\Lambda}(s)$ can be continued analytically to the whole complex plane, except for a simple pole at $s = 1$. We keep the notation $\zeta_{A,\Lambda}(s)$ for the function obtained in this way.

Every fundamental domain $\Omega_{k,j}$ of $\zeta_{A,\Lambda}(s)$ contains two disjoint sub-domains $\Omega^{+}_{k,j}$ and $\Omega^{-}_{k,j}$ which are represented conformally by the function respectively onto the upper half plane and onto the lower half plane with some slits. If $j \neq 0$ then one of them has as boundary the curve $\Gamma_{k,j}$, namely that is $\Omega^{+}_{k,j}$ if $j < 0$ and $\Gamma_{k,j}$ is not an embraced curve or if $j > 0$ and $\Gamma_{k,j}$ is embraced and that domain is $\Omega^{-}_{k,j}$ in the alternative situation. If the line $\mathrm{Re}\, s = \sigma_0$ intersects the pre-image of the negative real half axis, then it will separate the domain bounded by $\Gamma_{k,j}$ into two sub-domains: $\Delta'_{k,j}$ and $\Delta''_{k,j}$, the first one bounded and the second unbounded. The denseness property into the whole complex plane of the image of the line $\operatorname{Re} s = \sigma_0$ requires that the closure of $\cup_{k,j}\ \zeta_{A,\Lambda}(\Delta'_{k,j})$ covers the negative real half axis. This property is as surprising as the denseness itself, since every domain $\zeta_{A,\Lambda}(\Delta'_{k,j})$ is bounded. Moreover, the sets $\cup_{k,j}\ \zeta_{A,\Lambda}(\Delta'_{k,j})$ and $\cap_{k,j}\zeta_{A,\Lambda}(\Delta''_{k,j})$ are disjoint and their union covers infinitely many times the whole complex plane.

In what follows we will deal with the particular case where $\zeta_{A,\Lambda}(s)$ can be written as an Euler product:

$$\zeta_{A,\Lambda}(s) = \prod_{p\in P}(1 - a_p e^{-\lambda_p s})^{-1}, \tag{3.3}$$

where P is the set of prime numbers. With $\sigma > \sigma_a$, one can reproduce the Bohr construction (see [1]) for $\zeta_{A,\Lambda}(s)$ instead of $\zeta(s)$. Namely, given an arbitrary complex number $z \neq 0$, take a circle (K) passing through z and 1 of radius big enough and choose recursively the angles φ_n such that $z_n = z_{n-1}\left[1 + |a_{p_n}|e^{-\lambda_{p_n}\sigma}\cdot e^{i\varphi_n}\right] \in (K)$, where $z_1 = 1$, $\arg z_n > \arg z_{n-1}$ and p_n is the n-th prime number. Due to the absolute convergence of $\zeta_{A,\Lambda}(s)$ at $s = \sigma + it$, the sequence (z_n) converges and obviously

$$\lim_{n\to\infty} z_n = \prod_{n=1}^{\infty}\left[1 + |a_{p_n}|e^{-\lambda_{p_n}\sigma}\cdot e^{i\varphi_n}\right] \in (K). \tag{3.4}$$

As in [1], there are infinitely many ways of choosing $\sigma = \sigma_0$ such that $\lim_{n\to\infty} z_n = z$. It can be easily verified that (3.1) and (3.3) imply:

$$\sum_{n=1}^{\infty}|a_n|e^{-\lambda_n\sigma_0 + i\theta_n} = \prod_{p\in P}(1 - |a_p|e^{-\lambda_p\sigma_0 + i\theta_n})^{-1}, \tag{3.5}$$

where $\theta_n = \arg a_n$. If the numbers λ_n are such that any finite subset of Λ is linearly independent with respect to the rationals, then Diophantine approximation can be used for φ_{p_n} and $\mu_n = \pi - t\lambda_{p_n} + \theta_{p_n}$ and all the arguments used in [1] are valid for $\zeta_{A,\Lambda}(s)$ instead of $\zeta(s)$. Hence the following can be proved.

Theorem 3.1. *For ever $z \in \mathbb{C}\setminus\{0\}$ and every $\varepsilon > 0$ there is $\sigma_0 > \sigma_a$ and $t_0 \in \mathbb{R}$ such that*

$$\left|\frac{\zeta_{A,\Lambda}(\sigma_0 + it_0)}{z} - 1\right| < \varepsilon.$$

Denoting by $|A|$ the set $|a_1|, |a_2|, ...$ we have

$$\left|\sum_{n=1}^{\infty} a_n e^{-\lambda_n(\sigma_0 + it)}\right| \leq \sum_{n=1}^{\infty}|a_n|e^{-\lambda_n\sigma_0} = \zeta_{|A|,\Lambda}(\sigma_0),$$

hence $\zeta_{A,\Lambda}(\sigma_0 + it)$ is bounded on $\operatorname{Re} s = \sigma_0$. This shows that no line $\operatorname{Re} s = \sigma_0$ has the denseness property with respect to $\zeta_{A,\Lambda}(s)$ for $\sigma_0 > \sigma_a$, which generalizes Theorem 1.1.

If A is generated by a Dirichlet character and $\lambda_n = \log n$, then $\zeta_{A,\Lambda}(s)$ is an ordinary Dirichlet series and if $a_n = \chi(n)$, where χ is a Dirichlet character, we say that $\zeta_{A,\Lambda}(s)$ is a Dirichlet L-series. For such a series we have $\sigma_a = 1$. It is known that Dirichlet characters are totally multiplicative functions and the formula (3.3) takes place for any Dirichlet L-series. It is then expected that the image of a line $\operatorname{Re} s = \sigma_0$ by such a function be similar to the image of that line by the Riemann Zeta function. However, linear combinations of such functions are no longer Euler products and Bohr theory does not apply to them. Yet, the partition of the plane into S_k-strips and that of those strips into fundamental domains follows the same rules, and $\sigma_a = 1$ for each one of them, therefore the image of vertical lines by such a function should be similar to that produced by $\zeta(s)$. Moreover, for $\sigma_0 > 1$ and for a Dirichlet character χ modulo q we have

$$L(\chi, \sigma_0 + it) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{\sigma_0+it}} = \sum_{k=1}^{q} \sum_{n=1}^{\infty} \frac{\chi(nq+k)}{(nq+k)^{\sigma_0+it}} = \sum_{k=1}^{q} \chi(k) \sum_{n=1}^{\infty} \frac{1}{(nq+k)^{\sigma_0+it}}$$

and for $\sigma_0 > 1$ all these last q series are convergent, while for $\sigma_0 \leq 1$ at least one of them is divergent. The series of $L(\chi, \sigma_0 + it)$ has the same behavior, therefore the image by $L(\chi, s)$ of the line $\operatorname{Re} s = \sigma_0$, $\sigma_0 \leq 1$ is an unbounded set. In other words, from the point of view of the image of the line $\operatorname{Re} s = \sigma_0$, the functions $L(\chi, s)$ and $\zeta(s)$ behave similarly. This is obviously true also for linear combinations of Dirichlet L-series.

We have shown in [7] that the functions $\zeta_{A,\Lambda}(s)$ have at most one double zero in every strip S_k and no zero of higher order. The existence of a double zero has been revealed in [8]. Our computation reveals that for the function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(s) = 0.65697 \cdot f_0(s) + 0.34303 \cdot L(7, 4, s),$$

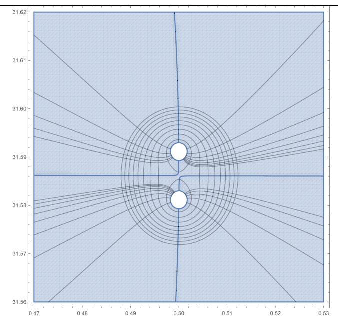
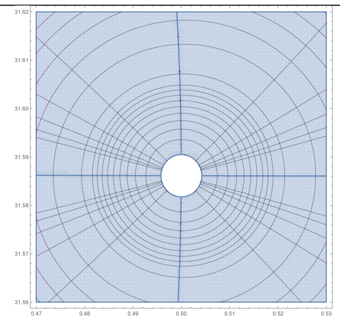
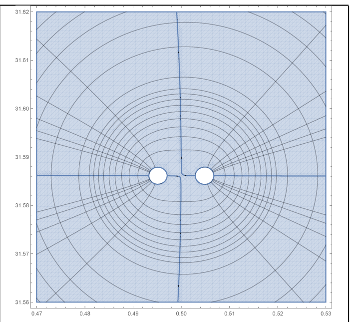
the double zero (see Table 1) is located at approximately

$$s_{\odot} = 0.5 + 31.586925 \cdot i.$$

Here we consider

$$f_0(s) = \frac{1}{2} \left\{ [L(7,2,s) + L(7,6,s)] + 0.6651818899 \cdot i \cdot [L(7,2,s) - L(7,6,s)] \right\}, \quad s \in \mathbb{C}.$$

Table 1. Plot of $(1-\tau) f_0(s) + \tau L(7,4,s)$ with specific values of τ

		
$\tau = 0.34306$	$\tau = 0.34303$	$\tau = 0.34300$

It is therefore of some interest to know the configuration of the image of a vertical line passing through that zero. Figs 6 and 7 from below represent the image of the segment of this line corresponding to $t \in [25, 40]$, respectively $t \in [30, 33]$.

4 Conclusions

A geometric approach to the study of Dirichlet series has been known for more than 80 years, yet it did not deal with denseness theorems. We have used in this paper modern computing tools in order to reveal geometric aspects of denseness theorems and color visualization in order to make our point of view more obvious. Our interest was limited to the results of the two pioneers in this field: H. Bohr and R. Courant.

Competing Interests

Authors have declared that no competing interests exist.

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