
**Journal of Advances in Mathematics and Computer Science**

**26(2): 1-22, 2018; Article no.JAMCS.39087**

***ISSN: 2456-9968***

*(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)*

# E-Bayesian Estimation of Two-Component Mixture of Inverse Lomax Distribution Based on Type-I Censoring Scheme

**Hesham M. Reyad[1*] and Soha A. Othman[2]**

[1]*College of Business and Economics, Qassim University, Kingdom of Saudi Arabia.*
[2]*Institute of Statistical Studies and Research, Cairo University, Egypt.*

***Authors' contributions***

*This work was carried out in collaboration among both authors. Author HMR introduced the idea in a methodically structure, did the data analysis and drafted the manuscript. Author SAO assisted in building the study design and also did the final proofreading. The two authors managed the analyses of the study and literature searches and approved the final manuscript.*

***Article Information***

DOI: 10.9734/JAMCS/2018/39087
Editor(s):
(1) Sheng Zhang, Professor, Department of Mathematics, Bohai University, Jinzhou, China.
Reviewers:
(1) Rashmi Awad, Devi Ahilya University, India.
(2) S. Zimeras, University of the Aegean, Greece.
(3) Thomas L. Toulias, Technological Educational Institute of Athens, Greece.
Complete Peer review History: http://www.sciencedomain.org/review-history/22993

***Original Research Article***

***Received: 19th November 2017***
***Accepted: 25th January 2018***
***Published: 5th February 2018***

## Abstract

This study is concerned with comparing the E-Bayesian and Bayesian methods for estimating the shape parameters of two-component mixture of inverse Lomax distribution based on type-i censored data. Based on the squared error loss (SELF), minimum expected loss (MELF), Degroot loss (DLF), precautionary loss (PLF), LINEX loss (LLF) and entropy loss (ELF) functions, formulas of E-Bayesian and Bayesian estimations are given. These estimates are derived based on a conjugate gamma prior and uniform hyperprior distributions. Comparisons among all estimates are performed in terms of absolute bias (ABias) and mean square error (MSE) via Monte Carlo simulation. Numerical computations showed that E-Bayesian estimates are more efficient than the corresponding Bayesian estimates.

*Keywords: Bayesian estimates; E-Bayesian estimates; inverse Lomax distribution; loss functions; mixture models.*

**Corresponding author: E-mail: hesham_reyad@yahoo.com;*

# 1 Introduction

Mixture models play a vital role in different applications such as cluster analysis, medicine, psychology, life testing and reliability analysis. A finite mixture of some probability distributions is advised to study a population that contain a number of subpopulations mixing in unknown proportions. Many statisticians have studied the mixture models for probability distributions; such as, Saleem et al. [1] introduced a Bayesian framework of two-component mixture of power function distribution. Kazmi et al. [2] constructed a Bayesian inference of two-component mixture of Maxwell distribution. Noor and Aslam [3] presented a Bayesian estimation of the inverse Weibull mixture model. Sultane et al. [4] discussed a Bayesian estimation of three-component mixture of Gumbel type-ii distribution.

The E-Bayesian estimation is a new method of estimation first pioneered by Han [5]. Han [6] derived the E-Bayesian and hierarchical Bayesian estimates of the reliability parameter of the exponential distribution under type-i censoring and by considering the quadratic loss function (QLF). Yin and Liu [7] obtained the E-Bayesian and hierarchical Bayesian estimates for the unknown reliability parameter of the geometric distribution based on scaled squared loss function (SSELF) in complete samples. Jaheen and Okasha [8] compared the Bayesian and E-Bayesian estimates for the parameters and reliability function of the Burr-xii distribution based on type-ii censoring and by considering the SELF and LLF. Azimi et al. [9] estimated the parameter and reliability function of the generalized half Logistic distribution by using the Bayesian and E-Bayesian methods under progressively type-ii censoring and by considering the SELF and LLF. Javadkani et al. [10] used the Bayesian, empirical Bayesian and E-Bayesian methods for estimating the unknown shape parameter and the reliability function of the two parameter bathtub-shaped lifetime distribution based on progressively first-failure-censored samples and by considering the MELF and LLF. Reyad and Othman [11] derived the Bayesian and E-Bayesian estimates for the shape parameter of the Gumbel type-ii distribution under type-ii censoring and by using SELF, LLF, DLF, QLF and MELF. Reyad and Othman [12] obtained the E-Bayesian and Bayesian estimates for the Kumaraswamy distribution under type-ii censored data and by using different symmetric and asymmetric loss functions. Reyad et al. [13] compared the E-Bayesian, hierarchical Bayesian, Bayesian and empirical Bayesian estimates of shape parameter and hazard function corresponding to the Gompertz distribution base on type-ii censoring and by using SELF, QLF, ELF and LLF. Reyad et al. [14] discussed the QE-Bayesian, quasi-Bayesian, quasi-hierarchical Bayesian and quasi-empirical Bayesian estimates for the scale parameter of the Erlang distribution under different loss functions in complete samples. Reyad et al. [15] compared the QE-Bayesian and E-Bayesian approaches for estimating scale parameter of the Frechet distribution based SELF, ELF, weighted balanced loss function (WBLF) and MELF in complete samples. Reyad et al. [16] compared the E-Bayesian and hierarchical Bayesian estimates of the scale parameter corresponding to the inverse Weibull distribution based on dual generalized order statistics based on various loss functions.

This paper aims to compare the E-Bayesian and Bayesian estimates of the shape parameters of two-component mixture of inverse Lomax distribution based on type-i censored data and different loss functions. A Monte Carlo simulation is used to assess the performance of all resulting estimates in terms of ABias and MSE.

The layout of the paper is organized as follow: In Section 2, the two-component mixture of inverse Lomax distribution is defined. In Section 3, the likelihood function under type-i censored is obtained. In Section 4, the Bayesian estimates of $\alpha_1$ and $\alpha_2$ under SELF, MELF, DLF, PLF,LLF and ELF are derived. In Section 5, the E-Bayesian estimates of $\alpha_1$ and $\alpha_2$ based on SELF, MELF, DLF, PLF,LLF and ELF are investigated. In Section 6, a Monte Carlo simulation is conducted to compare the efficiency of the resulting estimates. Some concluding remarks have been given in the last Section.

# 2 The Two-component Mixture of Inverse Lomax Distribution

The inverse Lomax distribution has probability density function (pdf) given by

$$f(x)=\frac{\alpha_i\beta_i}{x^2}\left(1+\frac{\beta_i}{x}\right)^{-(\alpha_i+1)}, \qquad x\geq 0,\ \ \alpha_i>0,\ \ \beta_i>0,\ i=1,2, \tag{1}$$

where $\alpha$ and $\beta$ are the shape and scale parameters respectively.

A density function for mixture of two components densities with unknown mixing weight $p$ is defined as follows:

$$f(x)=p\,f_1(x)+(1-p)f_2(x), \qquad 0<p<1. \tag{2}$$

Using (1) in (2), then the pdf of mixture of two density inverse Lomax is given by

$$f(x)=\frac{p\alpha_1\beta_1}{x^2}\left(1+\frac{\beta_1}{x}\right)^{-(\alpha_1+1)}+\frac{(1-p)\alpha_2\beta_2}{x^2}\left(1+\frac{\beta_2}{x}\right)^{-(\alpha_2+1)}. \tag{3}$$

The distribution function (cdf) corresponding to (3) is

$$F(x)=p\left(1+\frac{\beta_1}{x}\right)^{-\alpha_1}+(1-p)\left(1+\frac{\beta_2}{x}\right)^{-\alpha_2}. \tag{4}$$

# 3 The Sampling and Likelihood Function

Suppose $n$ units from two-component mixture of inverse Lomax distributions are used in a life testing experiment with a fixed test termination time $T$. Let $r$ units out of $n$ are failed until fixed test termination time $T$ and the remaining $(n-r)$ units are still working. Let $r_1$ and $r_2$ units out of $r$ units corresponding to subpopulation-I and subpopulation-II respectively such that $r=r_1+r_2$. Assume also that $x_{\ell k}, 0<x_{\ell k}<T$ be the failure time of the $k^{th}$ unit belonging to the $\ell^{th}$ subpopulation, where $\ell=1,2$ and $k=1,2,...,r_\ell$. (see Sultane et al. [4] page.288). The likelihood function in this case is given by

$$L(\alpha_1,\alpha_2,\beta_1,\beta_2,p\,|\,\underline{x})\propto\left(\prod_{j=1}^{r_1}p\,f_1(x_{1j})\right)\left(\prod_{j=1}^{r_2}(1-p)\,f_2(x_{2j})\right)\left(1-F(T)\right)^{n-r}. \tag{5}$$

Substituting from (3) and (4) in (5) and after some manipulations, we obtain

$$L(\alpha_1,\alpha_2,\beta_1,\beta_2,p\,|\,\underline{x})\propto\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}(-1)^k\,\alpha_1^{r_1}\,\alpha_2^{r_2}\,\beta_1^{r_1}\,\beta_2^{r_2}\,p^{r_1+k-m}\,(1-p)^{r_2+m}$$
$$\times\left(1+\frac{\beta_1}{T}\right)^{-\alpha_1(k-m)}\left(1+\frac{\beta_2}{T}\right)^{-\alpha_2 k}.$$

Suppose $\beta_1,\beta_2$ and $p$ are known, then the likelihood function is reduced to

$$L(\alpha_1,\alpha_2\,|\,\underline{x})\propto\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}(-1)^k\,\alpha_1^{r_1}\,\alpha_2^{r_2}\,e^{-(\alpha_1A+\alpha_2B)}, \tag{6}$$

where, $A=\sum_{j=1}^{r_1}\left(1+\frac{\beta_1}{x_{1j}}\right)+(k-m)\ln\left(1+\frac{\beta_1}{T}\right)$ and $B=\sum_{j=1}^{r_2}\left(1+\frac{\beta_2}{x_{2j}}\right)+k\ln\left(1+\frac{\beta_2}{T}\right)$.

# 4 Bayesian Estimation

In this section, we will obtain the Bayesian estimates of the shape parameters $\alpha_1$ and $\alpha_2$ of two-component mixture of inverse Lomax distribution by considering SELF, MELF, DLF, PLF, LLF and ELF.

Suppose that $\alpha_1$ and $\alpha_2$ have conjugated gamma prior distributions with pdfs given by

$$g_1(\alpha_1|a_1,b_1)=\frac{b_1^{a_1}}{\Gamma(a_1)}\,\alpha_1^{a_1-1}\,e^{-b_1\alpha_1},\qquad \alpha_1>0,\ a_1>0,\ b_1>0, \tag{7}$$

and

$$g_2(\alpha_2|a_2,b_2)=\frac{b_2^{a_2}}{\Gamma(a_2)}\,\alpha_2^{a_2-2}\,e^{-b_2\alpha_2},\qquad \alpha_2>0,\ a_2>0,\ b_2>0. \tag{8}$$

Then, the joint posterior distribution of $\alpha_1$ and $\alpha_2$ can be obtained by combining (6), (7) and (8) to be

$$h(\alpha_1,\alpha_2|\underline{x})=\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}(-1)^k\,\alpha_1^{r_1+a_1-1}\,\alpha_2^{r_2+a_2-1}\,e^{-\alpha_1(b_1+A)-\alpha_2(b_2+B)}, \tag{9}$$

where, $z=\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2}}$.

The marginal posterior distributions of $\alpha_1$ and $\alpha_2$ can be obtained from (9) to be

$$h_1(\alpha_1|\underline{x})=\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)}{(b_2+B)^{r_2+a_2}}\alpha_1^{r_1+a_1-1}\,e^{-\alpha_1(b_1+A)}, \tag{10}$$

and

$$h_2(\alpha_2|\underline{x})=\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)}{(b_1+A)^{r_1+a_1}}\,\alpha_2^{r_2+a_2-1}\,e^{-\alpha_2(b_2+B)}. \tag{11}$$

## 4.1 Bayesian estimation under SELF

Mood et al. [17] introduced the SELF as follows:

$$L_1(\hat{\alpha},\alpha)=a(\hat{\alpha}-\alpha)^2,\qquad a>0$$

where $\hat{\alpha}$ is an estimator of $\alpha$ and $a$ is the scale of the loss function. The scale $a$ is often taken equal to one which has no effect on the Bayesian estimates. This loss function is symmetric in nature. i.e. it gives equal importance to both over and under estimation. The Bayesian estimates of $\alpha_i\,(i=1,2)$ based on SELF denoted as $\hat{\alpha}_{iBS}\,(i=1,2)$ can be obtained as

$$\hat{\alpha}_{iBS} = E_{h_i}(\alpha_i \,|\, \underline{x}), \quad i = 1,2, \tag{12}$$

where $E_{h_i}(i=1,2)$ indicated to the expectation of the posterior distributions. We can obtain $\hat{\alpha}_{iBS}(i=1,2)$ by using (10) and (11) in (12) respectively to be

$$\hat{\alpha}_{1BS} = \left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1+1)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1+1}}, \tag{13}$$

and

$$\hat{\alpha}_{2BS} = \left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2+1)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2+1}}. \tag{14}$$

## 4.2 Bayesian estimation under MELF

The MELF is suggested by Tummala and Sathe [18] as

$$L_2(\hat{\alpha},\alpha) = \frac{(\hat{\alpha}-\alpha)^2}{\alpha^2}.$$

The Bayesian estimates of $\alpha_i(i=1,2)$ under MELF denoted as $\hat{\alpha}_{iBM}(i=1,2)$ can be calculated from

$$\hat{\alpha}_{iBM} = \frac{E_{h_i}(\alpha_i^{-1} \,|\, \underline{x})}{E_{h_i}(\alpha_i^{-2} \,|\, \underline{x})}, \qquad i = 1,2. \tag{15}$$

We can obtain $\hat{\alpha}_{iBM}(i=1,2)$ by using (10) and (11) in (15) respectively to be

$$\hat{\alpha}_{1BM} = \frac{\displaystyle\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1-1)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1-1}}}{\displaystyle\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1-2)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1-2}}}, \tag{16}$$

and

$$\hat{\alpha}_{2BM} = \frac{\displaystyle\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2-1)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2-1}}}{\displaystyle\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2-2)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2-2}}}. \tag{17}$$

## 4.3 Bayesian estimation under DLF

Degroot [19] defined the DLF as

$$L_3(\hat{\alpha},\alpha) = \frac{(\alpha-\hat{\alpha})^2}{\hat{\alpha}}.$$

The Bayesian estimates of $\alpha_i (i=1,2)$ based on DLF denoted as $\hat{\alpha}_{iBD}(i=1,2)$ can be derived from

$$\hat{\alpha}_{iBD} = \frac{E_{h_i}(\alpha_i^2 | \underline{x})}{E_{h_i}(\alpha_i | \underline{x})}, \qquad i = 1,2. \tag{18}$$

We can get $\hat{\alpha}_{iBD}(i=1,2)$ by using (10) and (11) in (18) respectively to be

$$\hat{\alpha}_{1BD} = \frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1+2)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1+2}}}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1+1)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1+1}}}, \tag{19}$$

and

$$\hat{\alpha}_{2BD} = \frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2+2)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2+2}}}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2+1)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2+1}}}. \tag{20}$$

## 4.4 Bayesian estimation under PLF

The PLF is proposed by Norstorm [20] as

$$L_4(\hat{\alpha},\alpha) = \frac{(\hat{\alpha}-\alpha)^2}{\hat{\alpha}}.$$

This loss function is approaches infinitely near the origin to prevent underestimation, thus giving conservative estimators, especially when low failure rates are being estimated. These estimates are very useful when underestimation may lead to serious consequences.

The Bayesian estimates of $\alpha_i (i=1,2)$ based on PLF denoted as $\hat{\alpha}_{iBP}(i=1,2)$ can be obtained as

$$\hat{\alpha}_{iBP} = \sqrt{E_{h_i}(\alpha_i^2 | \underline{x})}, \qquad i = 1,2. \tag{21}$$

We can calculate $\hat{\alpha}_{iBP}(i=1,2)$ by using (10) and (11) in (21) respectively to be

$$\hat{\alpha}_{1BP} = \left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1+2)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1+2}}\right]^{1/2} \tag{22}$$

and

$$\hat{\alpha}_{2BP} = \left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2+2)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2+2}}\right]^{1/2}. \tag{23}$$

## 4.5 Bayesian estimation under LLF

Zellner [21] represented the LLF as

$$L_5(\hat{\alpha},\alpha) = m\left\{\exp\left[w(\hat{\alpha}-\alpha)\right] - w(\hat{\alpha}-\alpha) - 1\right\},$$

with two parameters $m>0,\ w\neq 0$, where $m$ is the scale of the loss function and $w$ determines its shape. Without loss of generality, we assume $m=1$. The Bayesian estimates of $\alpha_i\,(i=1,2)$ relative to LLF denoted as $\hat{\alpha}_{iBL}\,(i=1,2)$ can be obtained as

$$\hat{\alpha}_{iBL}=\left(\frac{-1}{w}\right)\ln\left[E_{h_i}\left(e^{-w\alpha_i}\left|\underline{x}\right.\right)\right],\qquad i=1,2. \tag{24}$$

We can calculate $\hat{\alpha}_{iBL}\,(i=1,2)$ by using (10) and (11) in (24) respectively to be

$$\hat{\alpha}_{1BL}=\left(\frac{-1}{w}\right)\ln\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1)}{(b_2+B)^{r_2+a_2}\,(b_1+A+w)^{r_1+a_1}}\right], \tag{25}$$

and

$$\hat{\alpha}_{2BL}=\left(\frac{-1}{w}\right)\ln\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2)}{(b_1+A)^{r_1+a_1}\,(b_2+B+w)^{r_2+a_2}}\right]. \tag{26}$$

## 4.6 Bayesian estimation under ELF

Dey et al. [22] used the ELF of the form

$$L_6(\hat{\alpha},\alpha)\propto\left(\frac{\hat{\alpha}}{\alpha}\right)-\ln\left(\frac{\hat{\alpha}}{\alpha}\right)-1.$$

The Bayesian estimates of $\alpha_i\,(i=1,2)$ based on ELF denoted as $\hat{\alpha}_{iBE}\,(i=1,2)$ can be get from

$$\hat{\alpha}_{iBE}=\left[E_{h_i}(\alpha_i^{-1}\left|\underline{x}\right.)\right]^{-1},\qquad i=1,2. \tag{27}$$

We can obtain $\hat{\alpha}_{iBE}\,(i=1,2)$ by using (10) and (11) in (27) respectively to be

$$\hat{\alpha}_{1BE}=\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1-1)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1-1}}\right]^{-1}, \tag{28}$$

and

$$\hat{\alpha}_{2BE}=\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2-1)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2-1}}\right]^{-1}. \tag{29}$$

# 5 E-Bayesian Estimation

In this section, we will derive the E-Bayesian estimates of the shape parameters $\alpha_1$ and $\alpha_2$ of two-component mixture of inverse Lomax distribution based on SELF, MELF, DLF, PLF,LLF and ELF.

According to Han [23], the hyperparameters $a_i(i=1,2)$ and $b_i(i=1,2)$ must be selected to guarantee $g_i(\alpha_i|a_i,b_i), i=1,2$ given in (7) and (8) are decreasing functions of $\alpha_i(i=1,2)$. The derivative of $g_i(\alpha_i|a_i,b_i), i=1,2$ with respect to $\alpha_i(i=1,2)$ are given below

$$\frac{dg_i(\alpha_i|a_i,b_i)}{d\alpha_i} = \frac{b_i^{a_i}}{\Gamma(a_i)}\,\alpha_i^{a_i-2}\,e^{-b_i\alpha}\left[(a_i-1)-b_i\alpha_i\right], \qquad i=1,2. \tag{30}$$

Note that $a_i>0, b_i>0, i=1,2$ and $\alpha_i>0$ leads to $0<a_i<1, b_i>0,\ i=1,2$ due to $\frac{dg_i(\alpha_i|a_i,b_i)}{d\alpha_i}<0$, and therefore $g_i(\alpha_i|a_i,b_i),\ i=1,2$ are decreasing functions of $\alpha_i(i=1,2)$. We assume that $a_i(i=1,2)$ and $b_i(i=1,2)$ are independent with bivariate density functions

$$\pi_j(a_i,b_i)=\pi_j(a_i)\,\pi_j(b_i), \qquad j=1,2,3., \qquad i=1,2$$

Then, we have the following bivariate uniform hyperprior distributions:

$$\pi_1(a_1,b_1)=\frac{2(c_1-b_1)}{c_1^2}, \qquad 0<a_1<1,\ 0<b_1<c_1, \tag{31}$$

$$\pi_2(a_1,b_1)=\frac{1}{c_1}, \qquad 0<a_1<1,\ 0<b_1<c_1, \tag{32}$$

$$\pi_3(a_1,b_1)=\frac{2b_1}{c_1^2}, \qquad 0<a_1<1,\ 0<b_1<c_1, \tag{33}$$

$$\pi_4(a_2,b_2)=\frac{2(c_2-b_2)}{c_2^2}, \qquad 0<a_2<1,\ 0<b_2<c_2, \tag{34}$$

$$\pi_5(a_2,b_2)=\frac{1}{c_2}, \qquad 0<a_2<1,\ 0<b_2<c_2, \tag{35}$$

and

$$\pi_6(a_2,b_2)=\frac{2b_2}{c_2^2}, \qquad 0<a_2<1,\ 0<b_2<c_2. \tag{36}$$

Consequently, the E-Bayesian estimates of $\alpha_i(i=1,2)$ can be obtained from

$$\hat{\alpha}_{iEB}=E_{\pi_j}\left(\hat{\alpha}_{iB}(a_i,b_i)\right)=\iint_{\Omega}\hat{\alpha}_{iB}(a_i,b_i)\,\pi_j(a_i,b_i)\,da_i\,db_i, \qquad j=1,2,...,6,\ \ i=1,2, \tag{37}$$

where $E_{\pi_j}(j=1,2,...,6)$ stands for the expectation of the bivariate hyperprior distributions and $\hat{\alpha}_{iB}(a_i,b_i)$ are the Bayesian estimates of $\alpha_i(i=1,2)$ based on SELF, MELF, DLF, PLF,LLF and ELF.

## 5.1 E-Bayesian estimation under SELF

We can obtain the E-Bayesian estimate of $\alpha_1$ relative to SELF based on $\pi_1(a_1,b_1)$ which is denoted as $\hat{\alpha}_{1EBS1}$ by using (13) and (31) in (37) to be

$$\hat{\alpha}_{1EBS1} = \left(\frac{2}{c_1^2}\right)\int_0^1\int_0^{c_1}\left(\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\,\Gamma(r_1+a_1+1)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1+1}}\right)(c_1-b_1)\,db_1\,da_1. \quad (38)$$

Also, we can derive the E-Bayesian estimates of $\alpha_1$ relative to SELF based on $\pi_2(a_1,b_1)$ and $\pi_3(a_1,b_1)$ which are denoted as $\hat{\alpha}_{1EBS2}$ and $\hat{\alpha}_{1EBS3}$ by using (13), (32) in (37) and (13), (33) in (37) respectively to be

$$\hat{\alpha}_{1EBS2} = \left(\frac{1}{c_1}\right)\int_0^1\int_0^{c_1}\left(\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1+1)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1+1}}\right)db_1\,da_1, \quad (39)$$

and

$$\hat{\alpha}_{1EBS3} = \left(\frac{2}{c_1^2}\right)\int_0^1\int_0^{c_1}\left(\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1+1)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1+1}}\right)b_1\,db_1\,da_1. \quad (40)$$

Similarly, we can get the E-Bayesian estimates $\alpha_2$ relative to SELF based on $\pi_j(a_2,b_2),\ j=4,5,6$ which are denoted as $\hat{\alpha}_{2EBS\ell},\ \ell=1,2,3$ by using (14), (34) in (37), (14), (35), in (37) and (14), (36) in (37) respectively to be

$$\hat{\alpha}_{2EBS1} = \left(\frac{2}{c_2^2}\right)\int_0^1\int_0^{c_2}\left(\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2+1)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2+1}}\right)(c_2-b_2)\,db_2\,da_2, \quad (41)$$

$$\hat{\alpha}_{2EBS2} = \left(\frac{1}{c_2}\right)\int_0^1\int_0^{c_2}\left(\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2+1)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2+1}}\right)db_2\,da_2, \quad (42)$$

and

$$\hat{\alpha}_{2EBS3} = \left(\frac{2}{c_2^2}\right)\int_0^1\int_0^{c_2}\left(\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2+1)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2+1}}\right)b_2\,db_2\,da_2. \quad (43)$$

## 5.2 E-Bayesian estimation under MELF

We can derive the E-Bayesian estimate of $\alpha_1$ relative to MELF based on $\pi_1(a_1,b_1)$ which is denoted as $\hat{\alpha}_{1EBM1}$ by using (16) and (31) in (37) to be

$$\hat{\alpha}_{1EBM1} = \left(\frac{2}{c_1^2}\right)\int_0^1\int_0^{c_1}\left(\frac{\displaystyle\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1-1)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1-1}}}{\displaystyle\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1-2)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1-2}}}\right)(c_1-b_1)\,db_1\,da_1. \quad (44)$$

Also, we can get the E-Bayesian estimates of $\alpha_1$ relative to MELF based on $\pi_2(a_1, b_1)$ and $\pi_3(a_1, b_1)$ which are denoted as $\hat{\alpha}_{1EBM2}$ and $\hat{\alpha}_{1EBM3}$ by using (16), (32) in (37) and (16), (33) in (37) respectively to be

$$\hat{\alpha}_{1EBM2} = \left(\frac{1}{c_1}\right)\int_0^1\int_0^{c_1}\left(\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1-1)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1-1}}}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1-2)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1-2}}}\right)db_1\,da_1, \tag{45}$$

and

$$\hat{\alpha}_{1EBM3} = \left(\frac{2}{c_1^2}\right)\int_0^1\int_0^{c_1}\left(\frac{\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1-1)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1-1}}}{\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1-2)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1-2}}}\right)b_1\,db_1\,da_1. \tag{46}$$

Moreover, we can calculate the E-Bayesian estimates $\alpha_2$ relative to MELF based on $\pi_j(a_2, b_2),\ j = 4,5,6$ which are denoted as $\hat{\alpha}_{2EBM\ell},\ \ell = 1,2,3$ by using (17), (34) in (37), (17), (35), in (37) and (17), (36) in (37) respectively to be

$$\hat{\alpha}_{2EBM1} = \left(\frac{2}{c_2^2}\right)\int_0^1\int_0^{c_2}\left(\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2-1)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2-1}}}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2-2)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2-2}}}\right)(c_2-b_2)\,db_2\,da_2, \tag{47}$$

$$\hat{\alpha}_{2EBM2} = \left(\frac{1}{c_2}\right)\int_0^1\int_0^{c_2}\left(\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2-1)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2-1}}}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2-2)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2-2}}}\right)db_2\,da_2, \tag{48}$$

and

$$\hat{\alpha}_{2EBM3} = \left(\frac{2}{c_2^2}\right)\int_0^1\int_0^{c_2}\left(\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2-1)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2-1}}}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2-2)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2-2}}}\right)b_2\,db_2\,da_2. \tag{49}$$

## 5.3 E-Bayesian estimation under DLF

The E-Bayesian estimate of $\alpha_1$ relative to DLF based on $\pi_1(a_1, b_1)$ denoted as $\hat{\alpha}_{1EBD1}$ can be obtained by using (19) and (31) in (37) to be

$$\hat{\alpha}_{1EBD1} = \left(\frac{2}{c_1^2}\right)\int_0^1\int_0^{c_1}\left(\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\,\Gamma(r_1+a_1+2)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1+2}}}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\,\Gamma(r_1+a_1+1)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1+1}}}\right)(c_1-b_1)\,db_1\,da_1. \tag{50}$$

In addition, we can calculate the E-Bayesian estimates of $\alpha_1$ relative to DLF based on $\pi_2(a_1,b_1)$ and $\pi_3(a_1,b_1)$ which are denoted as $\hat{\alpha}_{1EBD2}$ and $\hat{\alpha}_{1EBD3}$ by using (19), (32) in (37) and (19), (33) in (37) respectively to be

$$\hat{\alpha}_{1EBD2} = \left(\frac{1}{c_1}\right)\int_0^1\int_0^{c_1}\left(\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\,\Gamma(r_1+a_1+2)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1+2}}}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\,\Gamma(r_1+a_1+1)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1+1}}}\right)db_1\,da_1, \tag{51}$$

and

$$\hat{\alpha}_{1EBD3} = \left(\frac{2}{c_1^2}\right)\int_0^1\int_0^{c_1}\left(\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1+2)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1+2}}}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1+1)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1+1}}}\right)b_1\,db_1\,da_1. \tag{52}$$

Furthermore, we can derive the E-Bayesian estimates $\alpha_2$ relative to DLF based on $\pi_j(a_2,b_2)$, $j=4,5,6$ which are denoted as $\hat{\alpha}_{2EBD\ell}$, $\ell=1,2,3$ by using (20), (34) in (37), (20), (35), in (37) and (20), (36) in (37) respectively to be

$$\hat{\alpha}_{2EBD1} = \left(\frac{2}{c_2^2}\right)\int_0^1\int_0^{c_2}\left(\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\,\Gamma(r_2+a_2+2)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2+2}}}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\,\Gamma(r_2+a_2+1)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2+1}}}\right)(c_2-b_2)\,db_2\,da_2, \tag{53}$$

$$\hat{\alpha}_{2EBD2} = \left(\frac{1}{c_2}\right)\int_0^1\int_0^{c_2}\left(\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\,\Gamma(r_2+a_2+2)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2+2}}}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\,\Gamma(r_2+a_2+1)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2+1}}}\right)db_2\,da_2, \tag{54}$$

and

$$\hat{\alpha}_{2EBD3} = \left(\frac{2}{c_2^2}\right)\int_0^1\int_0^{c_2}\left(\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\,\Gamma(r_2+a_2+2)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2+2}}}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\,\Gamma(r_2+a_2+1)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2+1}}}\right)b_2\,db_2\,da_2. \tag{55}$$

## 5.4 E-Bayesian estimation under PLF

The E-Bayesian estimate of $\alpha_1$ relative to PLF based on $\pi_1(a_1,b_1)$ denoted as $\hat{\alpha}_{1EBP1}$ can be calculated by using (22) and (31) in (37) to be

$$\hat{\alpha}_{1EBP1} = \left(\frac{2}{c_1^2}\right)\int_0^1\int_0^{c_1}\left(\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1+2)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1+2}}\right]^{1/2}\right)(c_1-b_1)\,db_1\,da_1. \quad (56)$$

Also, we can get the E-Bayesian estimates of $\alpha_1$ relative to PLF based on $\pi_2(a_1,b_1)$ and $\pi_3(a_1,b_1)$ which are denoted as $\hat{\alpha}_{1EBP2}$ and $\hat{\alpha}_{1EBP3}$ by using (22), (32) in (37) and (22), (33) in (37) respectively to be

$$\hat{\alpha}_{1EBP2} = \left(\frac{1}{c_1}\right)\int_0^1\int_0^{c_1}\left(\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1+2)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1+2}}\right]^{1/2}\right)db_1\,da_1, \quad (57)$$

and

$$\hat{\alpha}_{1EBP3} = \left(\frac{2}{c_1^2}\right)\int_0^1\int_0^{c_1}\left(\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1+2)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1+2}}\right]^{1/2}\right)b_1\,db_1\,da_1. \quad (58)$$

Also, we can obtain the E-Bayesian estimates $\alpha_2$ relative to MELF based on $\pi_j(a_2,b_2),\ j=4,5,6$ which are denoted as $\hat{\alpha}_{2EBM\ell},\ \ell=1,2,3$ by using (23), (34) in (37), (23), (35), in (37) and (23), (36) in (37) respectively to be

$$\hat{\alpha}_{2EBP1} = \left(\frac{2}{c_2^2}\right)\int_0^1\int_0^{c_2}\left(\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2+2)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2+2}}\right]^{1/2}\right)(c_2-b_2)\,db_2\,da_2, \quad (59)$$

$$\hat{\alpha}_{2EBP2} = \left(\frac{1}{c_2}\right)\int_0^1\int_0^{c_2}\left(\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2+2)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2+2}}\right]^{1/2}\right)db_2\,da_2, \quad (60)$$

and

$$\hat{\alpha}_{2EBP3} = \left(\frac{2}{c_2^2}\right)\int_0^1\int_0^{c_2}\left(\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2+2)}{(b_1+A)^{r_1+a_1}\,(b_2+B)^{r_2+a_2+2}}\right]^{1/2}\right)b_2\,db_2\,da_2. \quad (61)$$

## 5.5 E-Bayesian estimation under LLF

The E-Bayesian estimate of $\alpha_1$ relative to LLF based on $\pi_1(a_1,b_1)$ denoted as $\hat{\alpha}_{1EBL1}$ can be calculated by using (25) and (31) in (37) to be

$$\hat{\alpha}_{1EBL1} = \left(\frac{-2}{wc_1^2}\right)\int_0^1\int_0^{c_1}\left(\ln\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1)}{(b_2+B)^{r_2+a_2}\,(b_1+A+w)^{r_1+a_1}}\right]\right)(c_1-b_1)\,db_1\,da_1. \quad (62)$$

Similarly, we can get the E-Bayesian estimates of $\alpha_1$ relative to LLF based on $\pi_2(a_1,b_1)$ and $\pi_3(a_1,b_1)$ which are denoted as $\hat{\alpha}_{1EBL2}$ and $\hat{\alpha}_{1EBL3}$ by using (25), (32) in (37) and (25), (33) in (37) respectively to be

$$\hat{\alpha}_{1EBP2} = \left(\frac{-1}{wc_1}\right)\int_0^1\int_0^{c_1}\left(\ln\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1)}{(b_2+B)^{r_2+a_2}\,(b_1+A+w)^{r_1+a_1}}\right]\right)db_1\,da_1, \quad (63)$$

and

$$\hat{\alpha}_{1EBL3} = \left(\frac{-2}{wc_1^2}\right)\int_0^1\int_0^{c_1}\left(\ln\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1)}{(b_2+B)^{r_2+a_2}\,(b_1+A+w)^{r_1+a_1}}\right]\right)b_1\,db_1\,da_1. \quad (64)$$

Furthermore, we can derive the E-Bayesian estimates $\alpha_2$ relative to LLF based on $\pi_j(a_2,b_2),\ j=4,5,6$ which are denoted as $\hat{\alpha}_{2EBL\ell},\ \ell=1,2,3$ by using (26), (34) in (37), (26), (35), in (37) and (26), (36) in (37) respectively to be

$$\hat{\alpha}_{2EBL1} = \left(\frac{-2}{wc_2^2}\right)\int_0^1\int_0^{c_2}\left(\ln\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2)}{(b_1+A)^{r_1+a_1}\,(b_2+B+w)^{r_2+a_2}}\right]\right)(c_2-b_2)\,db_2\,da_2, \quad (65)$$

$$\hat{\alpha}_{2EBL2} = \left(\frac{-1}{wc_2}\right)\int_0^1\int_0^{c_2}\left(\ln\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2)}{(b_1+A)^{r_1+a_1}\,(b_2+B+w)^{r_2+a_2}}\right]\right)db_2\,da_2, \quad (66)$$

and

$$\hat{\alpha}_{2EBL3} = \left(\frac{-2}{wc_2^2}\right)\int_0^1\int_0^{c_2}\left(\ln\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2)}{(b_1+A)^{r_1+a_1}\,(b_2+B+w)^{r_2+a_2}}\right]\right)b_2\,db_2\,da_2. \quad (67)$$

## 5.6 E-Bayesian estimation under ELF

We can derive the E-Bayesian estimate of $\alpha_1$ relative to ELF based on $\pi_1(a_1,b_1)$ which is denoted as $\hat{\alpha}_{1EBE1}$ by using (28) and (31) in (37) to be

$$\hat{\alpha}_{1EBE1} = \left(\frac{2}{c_1^2}\right)\int_0^1\int_0^{c_1}\left(\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1-1)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1-1}}\right]^{-1}\right)(c_1-b_1)\,db_1\,da_1. \quad (68)$$

Also, we can get the E-Bayesian estimates of $\alpha_1$ relative to ELF based on $\pi_2(a_1,b_1)$ and $\pi_3(a_1,b_1)$ which are denoted as $\hat{\alpha}_{1EBE2}$ and $\hat{\alpha}_{1EBE3}$ by using (28), (32) in (37) and (28), (33) in (37) respectively to be

$$\hat{\alpha}_{1EBE2} = \left(\frac{1}{c_1}\right)\int_0^1\int_0^{c_1}\left(\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1-1)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1-1}}\right]^{-1}\right)db_1\,da_1, \quad (69)$$

and

$$\hat{\alpha}_{1EBE3} = \left(\frac{2}{c_1^2}\right)\int_0^1\int_0^{c_1}\left(\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_2+a_2)\Gamma(r_1+a_1-1)}{(b_2+B)^{r_2+a_2}\,(b_1+A)^{r_1+a_1-1}}\right]^{-1}\right)b_1\,db_1\,da_1. \quad (70)$$

Moreover, we can calculate the E-Bayesian estimates $\alpha_2$ relative to ELF based on $\pi_j(a_2,b_2),\ j=4,5,6$ which are denoted as $\hat{\alpha}_{2EBE\ell},\ \ell=1,2,3$ by using (29), (34) in (37), (29), (35), in (37) and (29), (36) in (37) respectively to be

$$\hat{\alpha}_{2EBE1} = \left(\frac{2}{c_2^2}\right)\int_0^1\int_0^{c_2}\left(\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2-1)}{(b_1+A)^{r_1+a_1}(b_2+B)^{r_2+a_2-1}}\right]^{-1}\right)(c_2-b_2)\,db_2\,da_2, \tag{71}$$

$$\hat{\alpha}_{2EBE2} = \left(\frac{1}{c_2}\right)\int_0^1\int_0^{c_2}\left(\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2-1)}{(b_1+A)^{r_1+a_1}(b_2+B)^{r_2+a_2-1}}\right]^{-1}\right)db_2\,da_2, \tag{72}$$

and

$$\hat{\alpha}_{2EBE3} = \left(\frac{2}{c_2^2}\right)\int_0^1\int_0^{c_2}\left(\left[\left(\frac{1}{z}\right)\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}\frac{(-1)^k\,\Gamma(r_1+a_1)\Gamma(r_2+a_2-1)}{(b_1+A)^{r_1+a_1}(b_2+B)^{r_2+a_2-1}}\right]^{-1}\right)b_2\,db_2\,da_2. \tag{73}$$

# 6 Monte Carlo Simulation

In this section, a Monte Carlo simulation study is carried out to evaluate the performance of the Bayesian and E-Bayesian estimates for the shape parameters associated to the two-component mixture of inverse Lomax distributions based on SELF, MELF, DLF, PLF,LLF and ELF described in the preceding sections. The simulation structure can be summarized in the following steps:

***Step (1):*** Set the default values (true values) of $\beta_1$, $\beta_2$, $c_1$, $c_2$, $w$ and $p$ which are 2, 3, 6, 4, -0.5 and 0.6 respectively. We considered different sample sizes ( $n = 25, 50, 75$ ) and test termination times ( $T = 20, 25$ ) to observe their effect on the resulting estimates

***Step (2):*** We generate $a_1$ and $b_1$ from the bivariate uniform hyperprior distributions; $\pi_i(a_1, b_1), i = 1,2,3$ given in (31), (32) and (33). For given values of $a_1$ and $b_1$ we generate $\alpha_1$ from the gamma prior distribution; $g_1(\alpha_1 | a_1, b_1)$ given in (7).

***Step (3):*** We generate $a_2$ and $b_2$ from the bivariate uniform hyperprior distributions; $\pi_i(a_2, b_2), i = 4,5,6$ given in (34), (35) and (36). For given values of $a_2$ and $b_2$ we generate $\alpha_2$ from the gamma prior distribution; $g_2(\alpha_2 | a_2, b_2)$ given in (8).

***Step (4):*** For known values of $\beta_1$, $\beta_2$, and $p$, type-i censored samples are generated from the two-component mixture of inverse Lomax distributions given in (3).

***Step (5):*** Calculate the Bayesian and E-Bayesian estimates of the unknown shape parameters associated to the two-component mixture of inverse Lomax distributions according to the formulas that have been obtained.

***Step (6):*** We repeated this process 1000 times and compute the absolute bias (ABias) and mean square error (MSE) for all estimates for different sample sizes, test termination times and given values of $\beta_1$, $\beta_2$, $c_1$, $c_2$, $w$ and $p$

where,

$$ABias(\hat{\alpha}) = |\hat{\alpha} - \alpha|, \qquad MSE(\hat{\alpha}) = \frac{1}{1000}\sum(\hat{\alpha} - \alpha)^2.$$

and $\hat{\alpha}$ stands for an estimator of $\alpha$. The simulation results are displayed in Tables (1-6).

**Table 1. Averaged values of ABias and MSEs (within parenthesis) for the Bayesian and E-Bayesian estimates of $\alpha_1$ and $\alpha_2$ based on SELF**

| $T$ | $n$ | Bayesian estimation | | E-Bayesian estimation | |
|---|---|---|---|---|---|
| | | $\hat{\alpha}_{1BS}$ | $\hat{\alpha}_{2BS}$ | $\hat{\alpha}_{1EBS}$ | $\hat{\alpha}_{2EBS}$ |
| | | | | 0.9718193 (0.9726478) | 0.1855762 (0.0568182) |
| 20 | 25 | 1.1251548 (1.2725078) | 0.3478656 (0.1242237) | 1.0620825 (1.1409433) | 0.1375378 (0.0279738) |
| | | | | 1.1523457 (1.3315204) | 0.4606518 (0.2139173) |
| | | | | 1.0192907 (1.0576375) | 0.201444 (0.051824) |
| 20 | 50 | 1.1436133 (1.313351) | 0.3159202 (0.1012314) | 1.0926467 (1.2029943) | 0.1208053 (0.0190389) |
| | | | | 1.1660026 (1.3625682) | 0.4430545 (0.1970497) |
| | | | | 1.0386734 (1.0910913) | 0.2069095 (0.0500876) |
| 20 | 75 | 1.1508893 (1.3284307) | 0.3043115 (0.0934683) | 1.1048034 (1.2266897) | 0.1149645 (0.0160597) |
| | | | | 1.1709334 (1.3731929) | 0.4368385 (0.1912854) |
| | | | | 0.9725897 (0.9735845) | 0.2908399 (0.1013358) |
| 25 | 25 | 1.1254682 (1.2733678) | 0.3136946 (0.100551 | 1.0626527 (1.1420531) | 0.0718313 (0.011766) |
| | | | | 1.1527158 (1.3324659) | 0.4345025 (0.1899424) |
| | | | | 1.0116273 (1.0414907) | 0.3021996 (0.1000878) |
| 25 | 50 | 1.1391387 (1.3027495) | 0.2831914 (0.0811617) | 1.0871175 (1.1905329) | 0.058177 (0.0067728) |
| | | | | 1.1626078 (1.3544282) | 0.4185536 (0.1757139) |
| | | | | 1.0312043 (1.0777314) | 0.3098757 (0.1012243) |
| 25 | 75 | 1.1475446 (1.3212712) | 0.2707832 (0.0738659) | 1.0999021 (1.216861) | 0.0510427 (0.0046009) |
| | | | | 1.1686254 (1.3680166) | 0.4119611 (0.1700104) |

**Table 2. Averaged values of ABias and MSEs (within parenthesis) for the Bayesian and E-Bayesian estimates of $\alpha_1$ and $\alpha_2$ based on MELF**

| $T$ | $n$ | Bayesian Estimation | | E-Bayesian Estimation | |
|---|---|---|---|---|---|
| | | $\hat{\alpha}_{1BM}$ | $\hat{\alpha}_{2BM}$ | $\hat{\alpha}_{1EBM}$ | $\hat{\alpha}_{2EBM}$ |
| 20 | 25 | 1.1590825 (1.3483735) | 0.05158105 (0.2755131) | 1.1545596 (1.3382692)<br>1.1586736 (1.3474615)<br>1.1627877 (1.3567005) | 0.490945 (0.2507415)<br>0.5128343 (0.2723434)<br>0.5347237 (0.2949418) |
| 20 | 50 | 1.1591772 (1.3484828) | 0.4048458 (0.1664493) | 1.1571713 (1.3439889)<br>1.159055 (1.3482089)<br>1.1609388 (1.3524385) | 0.3887502 (0.1535719)<br>0.4040404 (0.1657673)<br>0.4193306 (0.1784385) |
| 20 | 75 | 1.1610162 (1.3515039) | 0.364147 (0.1338037) | 1.1597631 (1.348668)<br>1.1609508 (1.3513557)<br>1.1621385 (1.3540469) | 0.3527433 (0.1255952)<br>0.3639133 (0.1336271)<br>0.3750833 (0.1419126) |
| 25 | 25 | 1.1590786 (1.3485225) | 0.4758405 (0.2334975) | 1.154598 (1.338512)<br>1.1586791 (1.3476316)<br>1.1627601 (1.3567963) | 0.4492508 (0.2088038)<br>0.4727732 (0.2304371)<br>0.4962955 (0.2532032) |
| 25 | 50 | 1.5550842 (1.3386662) | 0.3694763 (0.1383055) | 1.1530095 (1.3340252)<br>1.1549575 (1.3383827)<br>1.1569055 (1.3427502) | 0.3528017 (0.1261475)<br>0.3686954 (0.1377032)<br>0.3845892 (0.1497748) |
| 25 | 75 | 1.1578472 (1.3446353) | 0.3279392 (0.1089025) | 1.1565411 (1.3417038)<br>1.1577791 (1.3444819)<br>1.1590141 (1.3472639) | 0.317063 (0.1012881)<br>0.3285851 (0.1087598)<br>0.3401072 (0.1165016) |

**Table 3. Averaged values of ABias and MSEs (within parenthesis) for the Bayesian and E-Bayesian estimates of $\alpha_1$ and $\alpha_2$ based on DLF**

| $T$ | $n$ | Bayesian Estimation | | E-Bayesian Estimation | |
|---|---|---|---|---|---|
| | | $\hat{\alpha}_{1BD}$ | $\hat{\alpha}_{2BD}$ | $\hat{\alpha}_{1EBD}$ | $\hat{\alpha}_{2EBD}$ |
| 20 | 25 | 1.1081911 (1.2355665) | 0.2643344 (0.071831) | 1.1025896 (1.223773)<br>1.1076987 (1.2345353)<br>1.1128078 (1.2453715) | 0.222336 (0.0514825)<br>0.2594051 (0.0691986)<br>0.5347237 (0.0898778) |
| 20 | 50 | 1.1358313 (1.2959883) | 0.2715703 (0.0748633) | 1.1336127 (1.2911424)<br>1.1356997 (1.2957009)<br>1.1377867 (1.3002711) | 0.2513337 (0.0642565)<br>0.2706486 (0.0743518)<br>0.4193306 (0.085219) |
| 20 | 75 | 1.1458259 (1.3169777) | 0.274444 (0.0760831) | 1.1444863 (1.319928)<br>1.1457576 (1.3168253)<br>1.1470288 (1.3196618) | 0.2612538 (0.0690185)<br>0.2742157 (0.0759554)<br>0.3750833 (0.0832362) |
| 25 | 25 | 1.1086631 (1.2367535) | 0.232813 (0.055489) | 1.1031374 (1.2250939)<br>1.1081852 (1.2357485)<br>1.1132329 (1.2464729) | 0.1901668 (0.0376163)<br>0.2279455 (0.0532242)<br>0.4962955 (0.0719177) |
| 25 | 50 | 1.1311659 (1.2850024) | 0.2401018 (0.0584251) | 1.1288699 (1.2799971)<br>1.1310294 (1.2847048)<br>1.1331888 (1.2894249) | 0.2194882 (0.0489598)<br>0.2392113 (0.0579929)<br>0.3845892 (0.0678335) |
| 25 | 75 | 1.1423933 (1.3096763) | 0.2418008 (0.0589475) | 1.1409998 (1.3065918)<br>1.1423224 (1.3095184)<br>1.1436442 (1.3124494) | 0.2283707 (0.0526371)<br>0.2415971 (0.0588476)<br>0.3401072 (0.06544165) |

**Table 4. Averaged values of ABias and MSEs (within parenthesis) for the Bayesian and E-Bayesian estimates of $\alpha_1$ and $\alpha_2$ based on PLF**

| $T$ | $n$ | Bayesian Estimation | | E-Bayesian Estimation | |
|---|---|---|---|---|---|
| | | $\hat{\alpha}_{1BSP}$ | $\hat{\alpha}_{2BP}$ | $\hat{\alpha}_{1EBP}$ | $\hat{\alpha}_{2EBP}$ |
| 20 | 25 | 1.1168145 (1.2542624) | 0.3076894 (0.0971938) | 1.0559847 (1.1291415) | 0.0733373 (0.0118809) |
| | | | | 1.0998165 (1.2182513) | 0.2414227 (0.0617631) |
| | | | | 1.1436483 (1.3125421) | 0.4095082 (0.169157) |
| 20 | 50 | 1.1397543 (1.3047242) | 0.2941558 (0.0877854) | 1.0926237 (1.2038255) | 0.0860902 (0.0107086) |
| | | | | 1.1270339 (1.2767693) | 0.2424043 (0.0605087) |
| | | | | 1.161444 (1.352825) | 0.3987183 (0.1596719) |
| 20 | 75 | 1.1483716 (1.322729) | 0.2895584 (0.0846534) | 1.1064692 (1.2310002) | 0.0907405 (0.010421) |
| | | | | 1.1371508 (1.29761) | 0.2428101 (0.0601012) |
| | | | | 1.1678324 (1.3665628) | 0.3948796 (0.156369) |
| 25 | 25 | 1.1172041 (1.2552819) | 0.2746405 (0.0770608) | 1.0565739 (1.1303672) | 0.0157682 (0.00472268) |
| | | | | 1.1003027 (1.2194055) | (0.2001847 (0.0423828) |
| | | | | 1.1440316 (1.3135512) | 0.3846011 (0.1488383) |
| 25 | 50 | 1.1351853 (1.2939318) | 0.2620126 (0.0695068) | 1.0867651 (1.1905379) | 0.0312744 (0.0034017) |
| | | | | 1.1221798 (1.2654422) | 0.2031302 (0.042508) |
| | | | | 1.1575946 (1.3435938) | 0.374986 (0.1410799) |
| 25 | 75 | 1.1449832 (1.3154987) | 0.2564532 (0.0662751) | 1.1015223 (1.2210982) | 0.0348152 (0.0026663) |
| | | | | 1.1333621 (1.2896515) | 0.2027219 (0.0418391) |
| | | | | 1.1652019 (1.3607808) | 0.3706286 (0.1376363) |

**Table 5. Averaged values of ABias and MSEs (within parenthesis) for the Bayesian and E-Bayesian estimates of $\alpha_1$ and $\alpha_2$ based on LLF**

| $T$ | $n$ | Bayesian Estimation | | E-Bayesian Estimation | |
|---|---|---|---|---|---|
| | | $\hat{\alpha}_{1BL}$ | $\hat{\alpha}_{2BL}$ | $\hat{\alpha}_{1EBL}$ | $\hat{\alpha}_{2EBL}$ |
| | | | | 0. 5945314 (0.4218738) | 0.4966517 (0.2604015) |
| 20 | 25 | 1.1239565 (1.2699245) | 0.3356365 (0.1157731) | 1.1080333 (1.2370891) | 0.3123854 (0.0997804) |
| | | | | 1.6215353 (2.6340558) | 1.1214226 (1.2586466) |
| | | | | 0.6666425 (0.4990265) | 0.5083885 (0.265215) |
| 20 | 50 | 1.1431074 (1.3122354) | 0.3090865 (0.0969357) | 1.1399517 (1.3065727) | 0.325604 (0.1071234) |
| | | | | 1.6132608 (2.60689) | 1.1595966 (1.3449787) |
| | | | | 0.6931828 (0.5185077) | 0.5125692 (0.2670658) |
| 20 | 75 | 1.150578 (1.3277331) | 0.2996272 (0.0906343) | 1.1510637 (1.3297983) | 0.3306563 (0.1100683) |
| | | | | 1.6089445 (2.5918027) | 1.1738817 (1.3781561) |
| | | | | 0.5958344 (0.4247288) | 0.5763554 (0.3408514) |
| 25 | 25 | 1.1242829 (1.2708094) | 0.30110985 (0.092705) | 1.1086392 (1.2385696) | 0.2791523 (0.0793569) |
| | | | | 1.621444 (2.6338947) | 1.13466 (1.2882663) |
| | | | | 0.6524599 (0.4764771) | 0.5839615 (0.3455959) |
| 25 | 50 | 1.1386153 (1.3015973) | 0.2761814 (0.0772211) | 1.1347959 (1.2943385) | 0.2940956 (0.0872607) |
| | | | | 1.6171319 (2.6191054) | 1.1721527 (1.3741723) |
| | | | | 0.6827158 (0.5092838) | 0.5895438 (0.3502603) |
| 25 | 75 | 1.1472208 (1.3205503) | 0.2659891 (0.0712873) | 1.1473407 (1.3219012) | 0.2986451 (0.0896472) |
| | | | | 1.6119656 (2.6019564) | 1.1868341 (1.4086822) |

**Table 6. Averaged values of ABias and MSEs (within parenthesis) for the Bayesian and E-Bayesian estimates of $\alpha_1$ and $\alpha_2$ based on ELF**

| $T$ | $n$ | Bayesian Estimation | | E-Bayesian Estimation | |
|---|---|---|---|---|---|
| | | $\hat{\alpha}_{1BE}$ | $\hat{\alpha}_{2BE}$ | $\hat{\alpha}_{1EBE}$ | $\hat{\alpha}_{2EBE}$ |
| 20 | 25 | 1.1421186 (1.3101102) | 0.4316973 (0.1920583) | 1.653216 (1.13608278)<br>1.0940391 (1.2055299)<br>1.0227566 (1.0636397) | 0.4841919 (0.2397864)<br>0.3282017 (0.1190169)<br>0.4606518 (0.0444269) |
| 20 | 50 | 1.1513952 (1.3308492) | 0.3603461 (0.131753) | 1.1711135 (1.3741822)<br>1.1053199 (1.229309)<br>1.0395262 (1.0957042) | 0.4149878 (0.1761082)<br>0.211481 (0.0501303)<br>0.4430545 (0.0144933) |
| 20 | 75 | 1.1559528 (1.3399395) | 0.3342128 (0.1127065) | 1.1736995 (1.3795085)<br>1.1119459 (1.2418104)<br>1.0501922 (1.1135391) | 0.3912057 (0.1552097)<br>0.1631204 (0.0298157)<br>0.4368385 (0.0207314) |
| 25 | 25 | 1.1242829 (1.2708094) | 0.3010985 (0.092705) | 0.5958344 (0.4247228)<br>1.1086392 (1.2385696)<br>1.621444 (2.6338947) | 0.5763554 (0.3408514)<br>0.2791523 (0.0793569)<br>1.1346613 (1.2882663) |
| 25 | 50 | 1.1471115 (1.3206374) | 0.3263166 (0.1077853) | 1.1679185 (1.3664849)<br>1.1002942 (1.2178141)<br>1.0326695 (1.0808721) | 0.388777 (0.1534214)<br>0.1528731 (0.0276202)<br>0.4185536 (0.0273496) |
| 25 | 75 | 1.1526959 (1.3329242) | 0.2997813 (0.0905166) | 1.1715325 (1.3746802)<br>1.1074887 (1.2327524)<br>1.0434449 (1.1011727) | 0.3610423 (0.1324246)<br>0.1001328 (0.0123959)<br>0.4119611 (0.0512643) |

# 7 Conclusion Remarks

The E-Bayesian and Bayesian estimates are compared for the shape parameters of two-component mixture of inverse Lomax distribution based on type-i censoring. Numerical computations showed that E-Bayesian estimates are performing better than Bayesian estimates for $\alpha_1$ under different sample sizes, test termination times and various loss functions except for $T = 25$ and $n = 75$ under MELF where Bayesian estimates are the best. Moreover, the E-Bayesian estimates for $\alpha_2$ are more efficient than Bayesian estimates in most cases except for LLF where the Bayesian estimates are the best. Furthermore, comparing the E-Bayesian estimates under different loss functions, we can conclude that the E-Bayesian estimates for $\alpha_1$ based on LLF are the most efficient, whereas the E-Bayesian estimates based on MELF are the least efficient in all cases. On the other hand, the E-Bayesian estimates for $\alpha_2$ based on SELF are the best, whereas the E-Bayesian estimates based on LLF are the lowest in all cases. The ABias and MSE of all the resulting estimates decreases as the sample sizes and test termination times increases.

## Competing Interests

Authors have declared that no competing interests exist.

## References

[1] Saleem M, Aslam M, Economou P. On the Bayesian analysis of the mixture power function distribution using the complete and the censored sample. Journal of Applied Statistics. 2010; 37(1):25-40.

[2] Kazmi S, Aslam M, Ali S. On the Bayesian estimation for two component mixture of Maxwell distribution, assuming type-I censored data. International Journal of Applied Science and Technology. 2012;2(1):197-218.

[3] Noor F, Aslam M. Bayesian inference of the inverse Weibull distribution using type-I censoring. Journal of Applied Statistics. 2013;40(5):1076-1089.

[4] Sultana T, Aslam M, Raftab M. Bayesian estimation of 3-component mixture of Gumbel type-II distributions under non-informative and informative priors. J. Natn. Foundations Sri Lanka. 2017; 45(3):287-306.

[5] Han M. Expected Bayesian method for forecast of security investment. Journal of Operations Research and Management Science. 2005; 14(5): 89-102.

[6] Han M. E-Bayesian method to estimate failure rate. The Sixth International Symposium on Operations Research and Its Applications (ISOR06) Xinjiang. 2006;299-311.

[7] Yin Q, H. Liu H. Bayesian estimation of geometric distribution parameter under scaled squared error loss function. Conference on Environmental Science and Information Application Technology. 2010; 650-653.

[8] Jaheen ZF, Okasha HM. E-Bayesian estimation for the Burr type xii model based on type-2 censoring. Applied Mathematical Modelling. 2011;35:4730 - 4737.

[9] Azimi R, Yaghamei F, Fasihi B. E-Bayesian estimation based on generalized half Logistic progressive type-ii censored data. International Journal of Advanced Mathematical Science. 2013; 1(2):56-63.

[10] Javadkani N. Azhdari P. Azimi R. On Bayesian estimation from two parameter Bathtub-shaped lifetime distribution based on progressive first-failure-censored sampling. International Journal of Scientific World. 2014;2(1):31-41.

[11] Reyad, HM, Ahmed SO. E-Bayesian analysis of the Gumbel type-ii distribution under type-ii censored scheme. International Journal of Advanced Mathematical Sciences. 2015;3(2):108-120.

[12] Reyad, HM, Ahmed SO. Bayesian and E-Bayesian estimation for the Kumaraswamy distribution based on type-ii censoring. International Journal of Advanced Mathematical Sciences. 2016;4(1): 10-17.

[13] Reyad HM, Younis AM, Alkhedir AA. Comparison of Estimates using Censored Samples from Gompertz Model: Bayesian, E-Bayesian, Hierarchical Bayesian and Empirical Bayesian Schemes. International Journal of Advanced Statistics and Probability. 2016;4(1):47-61.

[14] Reyad HM, Younis AM, Alkhedir AA. Quasi-E-Bayesian criteria versus quasi-Bayesian, quasi-hierarchical Bayesian and quasi-empirical Bayesian methods for estimating the scale parameter of the Erlangen distribution. International Journal of Advanced Statistics and Probability. 2016;4(1): 62-74.

[15] Reyad HM, Younis AM, Alkhedir AA. QE-Bayesian and E-Bayesian approaches in estimation of scale parameter of the Frechet distribution. British Journal of Mathematics & Computer Science. 2016;19(2):1-29.

[16] Reyad HM, Younis AM, Ahmed SO. E-Bayesian and Hierarchical Bayesian Estimations Based on Dual Generalized Order Statistics from the Inverse Weibull Model. Journal of Advances in Mathematics and Computer Science. 2017;23(1):1-29.

[17] Mood A. Graybill FA, Boes D. Introduction to the theory of statistics. McGraw-Hill Series in Probability and Statistics; 1974.

[18] Tummala VM, Sathe PT. Minimum expected loss estimators of reliability and parameters of certain lifetime distributions. Reliability, IEEE Transactions. 1978;27(4):283-285.

[19] Degroot MH. Optimal statistical decision. McGraw-Hill Inc.; 1970.

[20] Nostrom JG. The use of precautionary loss function in risk analysis. IEEE Transaction on Reliability. 1996;45(3):400-403.

[21] Zellner A. Bayesian estimation and Prediction using Asymmetric loss Function. Journal of American Statistical Association. 1986;81:446-451.

[22] Dey DK, Gosh M, Srinivasan C. Simultaneous estimation of parameter under entropy loss. Journal of Statistical Planning and Inference. 1987;347-363.

[23] Han M. The structure of hierarchical prior distribution and its applications. Chinese Operations Research and Management Science. 1997;6(3):31-40.

---



***Peer-review history:***
*The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)*
*http://www.sciencedomain.org/review-history/22993*