Journal of Advances in Mathematics and Computer Science

26(2): 1-13, 2018; Article no.JAMCS.39023

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

The (P-A-L) Modified Weibull Distribution

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2018/39023
Editor(s):
(1) Andrej V. Plotnikov, Professor, Department of Applied and Calculus Mathematics and CAD, Odessa State Academy of Civil Engineering and Architecture, Ukraine.
Reviewers:
(1) Fellah Mamoun, Abbes Laghrour University and Annaba University, Algeria.
(2) Ramesh M. Mirajkar, Dr. Babasaheb Ambedkar College, India.
(3) Emrah Dokur, Bilecik S. E. University, Turkey.
Complete Peer review History: http://www.sciencedomain.org/review-history/22983

Received: 1st November 2017
Accepted: 20th January 2018
Published: 1st February 2018

Original Research Article

Abstract

This paper presented The (P-A-L) Modified Weibull Distribution. We can compute several properties of this distribution. The maximum likelihood estimators are obtained. Using simulation study, mean, relative bias, root and scaled mean square error for maximum likelihood estimators are obtained. And also, Confidence intervals for unknown two parameters are calculated.

Keywords: Hazard function; maximum likelihood estimation; modified weibull distribution; moment; the (P-A-L) family; variance-covariance matrix.

2010 Mathematics Subject Classification: 53C25, 83C05, 57N16.

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1 Introduction

Many companies and organizations have too much data and the amount of data available is growing. One of the ways to turn these data into useful information is to have a statistical model for the way the data are generated. On the other hand the industrial processes as well as the way the components of a business interact are getting more and more complicated. For these reasons, no matter how many distributional models become available there arise the needs for new models.

Several methods have been suggested in the statistical literature to construct new statistical distributions from existing ones. Among these methods there is the formula suggested by Marshal and Olkin [1] that became known by their name. There is also the rank transmutation map due to Shaw and Buckley [2]. Of interest to us here is the (P-A-L) extension suggested by Pappas et al. [3]. Lee et al. [4] present a good review of some of the most families methods of generating new families of distributions from existing ones.

In this article, the method known as the (P-A-L) is utilized to create a new family of distributions from the Modified Weibull distribution of Sarahan and Zaindin [5]. This distribution called The (P-A-L) Modified Weibull Distribution with four parameters α, β, γ, p where it denotes as PMWD $(\alpha, \beta, \gamma, p)$. We introduce PMWD $(\alpha, \beta, \gamma, p)$ in section 2 and present properties this distribution in section 3. We estimate four parameters for PMWD by the maximum likelihood estimation. Finally, we analyze real data for PMWD.

2 The (P-A-L) Modified Weibull Distribution

The survival function of the (P-A-L) family is

$$s(x) = \frac{\ln\{1 - (1-p)\, s_0(x)\}}{\ln p}, \quad x \in R_+,\ p \in R_+ - \{0\} \tag{2.1}$$

Where s_0 is the survival function of the base distribution and it should be noted that if $p \to 1$, then $s \to s_0$. The probability density function and hazard function take the form

$$f(x) = \frac{(p-1)\, f_0(x)}{\{1 - (1-p)\, s_0(x)\} \ln p} \tag{2.2}$$

$$h(x) = \frac{(p-1)\, s_0(x) h_0(x)}{\{1 - (1-p)\, s_0(x)\} \ln\{1 - (1-p)\, s_0(x)\}} \tag{2.3}$$

Where f_0 and h_0 are the probability density function and hazard function of the base distribution. There are several distributions used by the (P-A-L) family, such as the (P-A-L) extended modified Weibull studied by Pappas et al. [3], the (P-A-L) extended Weibull distribution introduced by Al-Zahrani et al. [6] and The (P-A-L) Generalized Exponential Distribution presented by Mahmoud and Mandouh [7].

Now, we put $s_0 = [e^{-\alpha x - \beta x^\gamma}]$ is a survival function for the Modified Weibull distribution presented by Sarahan and Zaindin [5].

By substituting s_0 in (2.1), we get

$$S(x) = \frac{\ln(1 - (1-p) e^{-\alpha x - \beta x^\gamma})}{\ln(p)} \quad ; \ x > 0,\ \gamma > 0,\ \alpha, \beta \geq 0,\ p \in R_+ - \{0\} \tag{2.4}$$

Hence

$$f(x) = \frac{(p-1)(\alpha+\beta\gamma x^{\gamma-1})e^{-\alpha x-\beta x^{\gamma}}}{(1-(1-p)e^{-\alpha x-\beta x^{\gamma}})\ln(p)} \tag{2.5}$$

Then the hazard function takes the form

$$h(x) = \frac{(p-1)e^{-\alpha x-\beta x^{\gamma}}[\alpha+\beta\gamma x^{\gamma-1}]}{[1-(1-p)e^{-\alpha x-\beta x^{\gamma}}]\ln(1-(1-p)e^{-\alpha x-\beta x^{\gamma}})} \tag{2.6}$$

We note that h(x) can be constant, increasing or decreasing depending on the parameter values. For example, if $p \to 1$, β=1 and γ=1 then h(x)=α+1 is constant, whereas if $p \to 1$ and β=1, then $h(x) = \alpha + \gamma x^{\gamma-1}$, which is increasing for $\gamma > 1$ and decreasing for $\gamma < 1$.

Figs. 1 and 2 show different shapes for the probability density function and the hazard function of The (P-A-L) Modified Weibull Distribution for different values of the parameters α, β, γ and p.

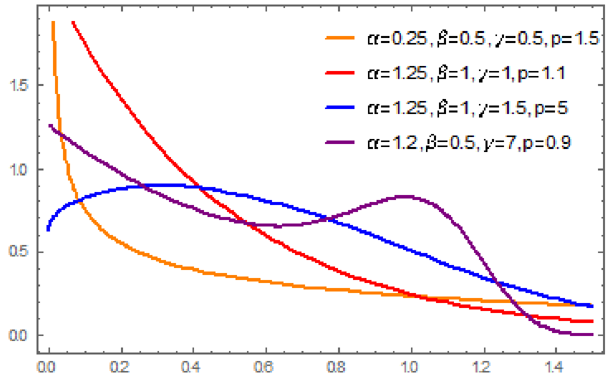

Fig. 1. Probability density function of the PAL Modified Weibull distribution for different values of the parameters

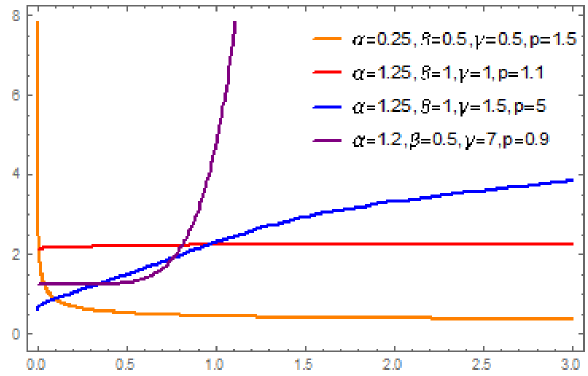

Fig. 2. Hazard function of the PAL Modified Weibull distribution for different values of the parameters

To find the raw moments, we have

$$E(X^r) = \int_0^\infty x^r f(x)dx$$

from equation(2.5), we get

$$E(X^r) = \int_0^\infty x^r \frac{(p-1)(\alpha+\beta\gamma x^{\gamma-1})e^{-\alpha x-\beta x^\gamma}}{(1-(1-p)e^{-\alpha x-\beta x^\gamma})\ln(1-(1-p))}dx \tag{2.7}$$

To calculate the r^{th} raw moments of The (P-A-L) Modified Weibull Distribution, we can use Numerical integration steps. We can use the Binomial expansion to get the following expression:

$$E(X^r) = \sum_{n=0}^{\infty} \frac{(-1)^n(p-1)^{(n+1)}}{log(1-(1-p))} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{(n+1)^i\beta^i}{(n+1)^{r+i\gamma}} \left[\frac{\Gamma(i\gamma+r+1)}{(n+1)\alpha^{i\gamma+r}} + \frac{\beta\gamma}{(n+1)^\gamma\alpha^{\gamma+i\gamma+r}}\Gamma(\gamma+i\gamma+k)\right] \tag{2.8}$$

Let $\alpha = 1.1, \beta = 2, \gamma = 3, p = 2$ and $r = 1$, one can easily check that equation (2.8) and numerical integration of (2.7) take the same value, i.e. $E(X^r) = 0.53$.

3 Random Number Generation and Estimation of the Parameters

Using the inversion method, one can generate random from The (P-A-L) Modified Weibull Distribution with the following formula

$$u = F(x) \tag{3.1}$$

By substituting F(x) in (3.1), we get

$$u = 1 - \frac{log(1-(1-p)e^{-\alpha x-\beta x^\gamma})}{log(p)} \tag{3.2}$$

Where $u \in (0, 1)$. We take log

$$\beta x^\gamma + \alpha x + log\frac{1-(1-(1-p))^{(1-u)}}{-(p-1)} = 0 \tag{3.3}$$

The equation (3.3) has no closed form solution in x_q so, we have to use numerical technique.

Now, we will study parameter estimation by using maximum likelihood.

3.1 Maximum likelihood estimation

Let $x_1, x_2, x_3, ..., x_n$ is a random sample follows The (P-A-L) Modified Weibull Distribution. The likelihood function is given by

$$L(x; \alpha, \beta, \gamma, p) = \Pi_{i=1}^n \frac{(p-1)(\alpha+\beta\gamma x^{\gamma-1})e^{-\alpha x-\beta x^\gamma}}{(1-(1-p)e^{-\alpha x-\beta x^\gamma})log(p)} \tag{3.4}$$

The log likelihood is

$$\ln L = n[ln(p-1)] + \sum_{i=1}^{n} ln(\alpha + \beta\gamma x_i^{\gamma-1}) + \sum_{i=1}^{n}(-\alpha x_i - \beta x_i^{\gamma})$$
$$-n[ln(ln(p))] - \sum_{i=1}^{n} ln(1-(1-p)e^{-\alpha x_i - \beta x_i^{\gamma}}) \tag{3.5}$$

Differentiating (3.5) with respect to α, β, γ and p, we have

$$\frac{\partial \ln L}{\partial \alpha} = \sum_{i=1}^{n} \frac{1}{\alpha + \beta\gamma x_i^{\gamma-1}} - \sum_{i=1}^{n} x_i + (p-1)\sum_{i=1}^{n} \frac{x_i e^{-\alpha x_i - \beta x_i^{\gamma}}}{1-(1-p)e^{-\alpha x_i - \beta x_i^{\gamma}}} \tag{3.6}$$

$$\frac{\partial \ln L}{\partial \beta} = \gamma\sum_{i=1}^{n} \frac{x_i^{\gamma-1}}{\alpha + \beta\gamma x_i^{\gamma-1}} - \sum_{i=1}^{n} x_i^{\gamma} + (p-1)\sum_{i=1}^{n} \frac{x_i^{\gamma} e^{-\alpha x_i - \beta x_i^{\gamma}}}{1-(1-p)e^{-\alpha x_i - \beta x_i^{\gamma}}} \tag{3.7}$$

$$\frac{\partial \ln L}{\partial \gamma} = \beta\sum_{i=1}^{n} \frac{x_i^{\gamma-1} + \gamma(\gamma-1)x_i^{\gamma-2}}{\alpha + \beta\gamma x_i^{\gamma-1}} - \beta\gamma\sum_{i=1}^{n} x_i^{\gamma-1} + (p-1)\beta\gamma\sum_{i=1}^{n} \frac{x_i^{\gamma-1} e^{-\alpha x_i - \beta x_i^{\gamma}}}{1+(p-1)e^{-\alpha x_i - \beta x_i^{\gamma}}} \tag{3.8}$$

$$\frac{\partial \ln L}{\partial p} = \frac{n}{(p-1)} - \frac{n}{\{1-(1-p)\}\{ln(1-(1-p))\}} - \sum_{i=1}^{n} \frac{e^{-\alpha x_i - \beta x_i^{\gamma}}}{1-(1-p)e^{-\alpha x_i - \beta x_i^{\gamma}}} \tag{3.9}$$

Equating the derivatives in (3.6), (3.7), (3.8) and (3.9) to zero, then we solve the four nonlinear equations by numerically, we get the maximum likelihood estimators $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{p}$.

Hence

The second derivatives of the log likelihood function are given by

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -I_{11} = -\Sigma_{i=1}^{n}\left(-\frac{e^{-2\alpha x_i - 2\beta x_i^{\gamma}}(p-1)^2 x_i^2}{(1+e^{-\alpha x_i - \beta x_i^{\gamma}}(p-1))^2} + \frac{e^{-\alpha x_i - \beta x_i^{\gamma}}(p-1)x_i^2}{1+e^{-\alpha x_i - \beta x_i^{\gamma}}(p-1)}\right)$$
$$+\Sigma_{i=1}^{n} - \frac{1}{(\alpha + \beta\gamma x_i^{\gamma-1})^2}$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -I_{22} = -\Sigma_{i=1}^{n} \frac{\gamma^2 x_i^{-2-2\gamma}}{(\alpha + \beta\gamma x_i^{-1+\gamma})^2} - -\Sigma_{i=1}^{n}\left(-\frac{e^{-2\alpha x_i - 2\beta x_i^{\gamma}}(p-1)^2 x_i^{2\gamma}}{(1+e^{-\alpha x_i - \beta x_i^{\gamma}}(p-1))^2}\right.$$
$$\left.+\frac{e^{-\alpha x_i - \beta x_i^{\gamma}}(p-1)x_i^{2\gamma}}{1+e^{-\alpha x_i - \beta x_i^{\gamma}}(p-1)}\right)$$

$$\frac{\partial^2 \ln L}{\partial \gamma^2} = -I_{33} = \Sigma_{i=1}^{n} - \beta\log[x_i]^2 x_i^{\gamma} + \Sigma_{i=1}^{n}[-\frac{(\beta x_i^{-1+\gamma} + \beta\gamma\log[x_i]x_i^{-1+\gamma})^2}{(\alpha + \beta\gamma x_i^{-1+\gamma})^2}$$
$$+\frac{2\beta\log[x_i]x_i^{-1+\gamma} + \beta\gamma\log[x_i]^2 x_i^{-1+\gamma}}{\alpha + \beta\gamma x_i^{-1+\gamma}}] - \Sigma_{i=1}^{n}[-\frac{e^{-\alpha x_i - \beta x_i^{\gamma}}\beta(p-1)\log[x_i]^2 x_i^{\gamma}}{1+e^{-\alpha x_i - \beta x_i^{\gamma}}(p-1)}$$

$$-\frac{e^{-2\alpha x_i-2\beta x_i^\gamma}\beta^2(p-1)^2\log[x_i]^2x_i^{2\gamma}}{(1+e^{-\alpha x_i-\beta x_i^\gamma}(p-1))^2}+\frac{e^{-\alpha x_i-\beta x_i^\gamma}\beta^2(p-1)\log[x_i]^2x_i^{2\gamma}}{1+e^{-\alpha x_i-\beta x_i^\gamma}(p-1)}]$$

$$\frac{\partial^2\ln L}{\partial p^2}=-I_{44}=-\frac{n}{(p-1)^2}+\frac{n}{(1-(1-p))^2\log[1-(1-p)]^2}+\frac{n}{(1-(1-p))^2Log[1-(1-p)]}$$

$$-\Sigma_{i=1}^n\frac{-e^{-2\alpha x_i-2\beta x_i^\gamma}}{(1+e^{-\alpha x_i-\beta x_i^\gamma}(p-1))^2}$$

$$\frac{\partial^2 lnL}{\partial\alpha\partial\beta}=-I_{12}=\Sigma_{i=1}^n\frac{-\gamma x_i^{-1+\gamma}}{(\alpha+\beta\gamma x_i^{-1+\gamma})^2}-\Sigma_{i=1}^n(-\frac{e^{-2\alpha x_i-2\beta x_i^\gamma}(p-1)^2x_i^{1+\gamma}}{(1+e^{-\alpha x_i-\beta x_i^\gamma}(p-1))^2}+\frac{e^{\alpha x_i-\beta x_i^\gamma}(p-1)x_i^{1+\gamma}}{1+e^{-\alpha x_i-\beta x_i^\gamma}(p-1)})$$

$$\frac{\partial^2 lnL}{\partial\alpha\partial\gamma}=-I_{13}=\Sigma_{i=1}^n-\frac{\beta x_i^{-1+\gamma}+\beta\gamma\log[x_i]x_i^{-1+\gamma}}{(\alpha+\beta\gamma x_i^{-1+\gamma})^2}-\Sigma_{i=1}^n[-\frac{e^{-2\alpha x_i-2\beta x_i^\gamma}\beta(p-1)^2\log[x_i]x_i^{1+\gamma}}{(1+e^{-\alpha x_i-\beta x_i^\gamma}(p-1))^2}$$

$$+\frac{e^{-\alpha x_i-\beta x_i^\gamma}\beta(p-1)\log[x_i]x_i^{1+\gamma}}{1+e^{-\alpha x_i-\beta x_i^\gamma}(p-1)}]$$

$$\frac{\partial^2 lnL}{\partial\alpha\partial p}=-I_{14}=-\Sigma_{i=1}^n(\frac{e^{-2\alpha x_i-2\beta x_i^\gamma}(p-1)x_i}{(1+e^{-\alpha x_i-\beta x_i^\gamma}(p-1))^2}-\frac{e^{-\alpha x_i-\beta x_i^\gamma}x_i}{1+e^{-\alpha x_i-\beta x_i^\gamma}(p-1)})$$

$$\frac{\partial^2 lnL}{\partial\beta\partial\gamma}=-I_{23}=\Sigma_{i=1}^n-\log[x_i]x_i^\gamma-\Sigma_{i=1}^n[-\frac{-e^{-\alpha x_i-\beta x_i^\gamma}(p-1)\log[x_i]x_i^\gamma}{1+e^{-\alpha x_i-\beta x_i^\gamma}(p-1)}$$

$$-\frac{e^{-2\alpha x_i-2\beta x_i^\gamma}\beta(p-1)^2\log[x_i]x_i^{2\gamma}}{(1+e^{-\alpha x_i-\beta x_i^\gamma}(p-1))^2}+\frac{-e^{-\alpha x_i-\beta x_i^\gamma}\beta(p-1)\log[x_i]x_i^{2\gamma}}{1+e^{-\alpha x_i-\beta x_i^\gamma}(p-1)}]+\Sigma_{i=1}^n[\frac{x_i^{-1+\gamma}}{\alpha+\beta\gamma x_i^{-1+\gamma}}$$

$$+\frac{\gamma\log[x_i]x_i^{-1+\gamma}}{\alpha+\beta\gamma x_i^{-1+\gamma}}-\frac{\gamma x_i^{-1+\gamma}(\beta x^{-1+\gamma}+\beta\gamma\log[x_i]x^{-1+\gamma})}{(\alpha+\beta\gamma x_i^{-1+\gamma})^2}]$$

$$\frac{\partial^2 lnL}{\partial\beta\partial p}=-I_{24}=-\Sigma_{i=1}^n(\frac{e^{-2\alpha x_i-2\beta x_i^\gamma}(p-1)x_i^\gamma}{(1+e^{-\alpha x_i-\beta x_i^\gamma}(p-1))^2}-\frac{e^{-\alpha x_i-\beta x_i^\gamma}x_i^\gamma}{1+e^{-\alpha x_i-\beta x_i^\gamma}(p-1)})$$

$$\frac{\partial^2 lnL}{\partial\gamma\partial p}=-I_{34}=-\Sigma_{i=1}^n(\frac{e^{-2\alpha x_i-2\beta x_i^\gamma}\beta(p-1)\log[x_i]x_i^\gamma}{(1+e^{-\alpha x_i-\beta x_i^\gamma}(p-1))^2}-\frac{e^{-\alpha x_i-\beta x_i^\gamma}\beta\log[x_i]x_i^\gamma}{1+e^{-\alpha x_i-\beta x_i^\gamma}(p-1)})$$

Then the observed information matrix is given by

$$I=\begin{pmatrix}I_{11}&I_{12}&I_{13}&I_{14}\\I_{21}&I_{22}&I_{23}&I_{24}\\I_{31}&I_{32}&I_{33}&I_{34}\\I_{41}&I_{42}&I_{43}&I_{44}\end{pmatrix}$$

So that the variance-covariance matrix may be approximated as

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & V_{24} \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & V_{42} & V_{43} & V_{44} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{pmatrix}^{-1}$$

It is known that the asymptotic distribution of the MLE $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{p})$ is given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\gamma} \\ \hat{p} \end{pmatrix} \sim N \left[\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ p \end{pmatrix}, \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & V_{24} \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & V_{42} & V_{43} & V_{44} \end{pmatrix} \right] \tag{3.10}$$

Since V involves the parameters α, β, γ, p, we replace the parameters by the corresponding MLE´ s in order to obtain an estimate of V, which is denoted by

$$\hat{V} = \begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{pmatrix}^{-1} \tag{3.11}$$

Where $\hat{I_{ij}}$ when $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{p})$ replaces $(\alpha, \beta, \gamma, p)$.
By using (3.10), approximate $100(1-\theta)\%$ confidence intervals for α, β, γ, p are determined, respectively, as

$$\hat{\alpha} \pm z_{\theta/2}\sqrt{\hat{V_{11}}}, \quad \hat{\beta} \pm z_{\theta/2}\sqrt{\hat{V_{22}}}, \quad \hat{\gamma} \pm z_{\theta/2}\sqrt{\hat{V_{33}}}, \quad \hat{p} \pm z_{\theta/2}\sqrt{\hat{V_{44}}} \tag{3.12}$$

Where z_θ is the upper θ-th percentile of the standard normal distribution.

4 Simulation Study

We used a simulation study to check the performance of the accuracy of point and interval estimates for Several cases, of which estimates two parameters, estimates three parameters and finally, estimates four parameters of $PMWD(\alpha, \beta, \gamma, p)$ for m=1000, the sample size n are 50, 100, 150, 200, 250, 300 and different parameter values. The following steps were followed to obtain the results:

1. Specify initial values for parameters α, β, γ and p.
2. Specify the sample size n.
3. Generate m times of random sample with size n from $PMWD(\alpha, \beta, \gamma, p)$.
4. Obtain the maximum likelihood estimates for α, β, γ, p for different sample sizes.
5. Obtain the mean, bias and relative bias, mean squared error (MSE), root mean squared error and scaled root mean square error for each estimator for different sample size.
6. Repeat 1:5 for several values of α, β, γ and p.

By the software Mathematica 10 we tried to do simulation and estimate the parameters (γ, p) with respect to (α, β) are known, The result is shown in Table 1 and we estimate the parameters (β, p) with respect to (α, γ) are known as shown in Table 2. Similarly, we estimate the parameters (β, γ, p) with respect to α are known as shown in Table 3. Finally, we estimate the parameters $(\alpha, \beta, \gamma, p)$ as shown in Table 4.

Based on the values we obtained from Simulation study, the results are shown in Tables 1, 2, 3 and 4.

Table 1. MLE's and confidence intervals of two parameter (γ and p) in the case of the (P-A-L) modified Weibull distribution

		Mean	Relative bias	variance	RMSE	Scaled RMSE	LL	UL
1. $\alpha_0 = 1.25, \beta_0 = 2.5, \gamma_0 = 1.5, p_0 = 1.98$								
n=50	γ	1.64	-0.09	0.25	0.52	0.35	1.03	2.25
	p	2.41	-0.22	3.52	1.92	0.97	-0.83	5.64
n=75	γ	1.60	-0.7	0.22	0.48	0.32	0.88	2.70
	p	2.38	-0.20	2.52	1.64	0.83	-2.47	7.24
n=100	γ	1.59	-0.06	0.18	0.44	0.27	1.08	0.39
	p	2.30	-0.16	1.96	1.44	0.62	2.10	4.20
n=150	γ	1.57	-0.05	0.12	0.35	0.23	1.18	1.96
	p	2.19	-0.11	1.26	1.14	0.52	0.81	3.58
2. $\alpha_0 = 1.25, \beta_0 = 2.5, \gamma_0 = 0.5, p_0 = 0.9$								
		Mean	Relative bias	variance	RMSE	Scaled RMSE	LL	UL
n=50	γ	0.54	-0.04	0.007	0.09	0.18	0.06	1.02
	p	0.98	-0.08	0.490	0.71	0.72	-3.47	5.45
n=75	γ	0.53	-0.06	0.005	0.08	0.15	0.41	0.65
	p	0.97	-0.08	0.350	0.60	0.62	-0.56	2.52
n=100	γ	0.52	-0.05	0.003	0.06	0.12	0.46	0.59
	p	0.94	-0.04	0.180	0.43	0.46	-0.05	1.92
n=150	γ	0.51	-0.03	0.002	0.05	0.09	0.29	0.75
	p	0.93	-0.04	0.130	0.36	0.39	-1.36	3.24
3. $\alpha_0 = 1.25, \beta_0 = 2.5, \gamma_0 = 7, p_0 = 5$								
		Mean	Relative bias	variance	RMSE	Scaled RMSE	LL	UL
n=50	γ	7.18	-0.03	02.87	1.71	0.23	4.24	10.13
	p	5.85	-0.17	10.25	3.31	0.56	0.41	11.31
n=75	γ	7.34	-0.04	02.45	1.60	0.22	4.76	09.93
	p	5.47	-0.09	06.85	2.66	0.48	0.89	10.05
n=100	γ	7.19	-0.02	01.69	1.32	0.18	2.63	11.76
	p	5.46	-0.09	05.87	2.46	0.45	-6.57	17.49
n=150	γ	7.09	-0.01	01.04	1.02	0.14	4.76	09.42
	p	5.26	-0.05	03.46	1.88	0.36	1.68	08.86

Table 2. MLE's and confidence intervals of two parameter (β and p) in the case of the (P-A-L) modified Weibull distribution

		Mean	Relative bias	variance	RMSE	Scaled RMSE	LL	UL
1. $\alpha_0 = 1.25, \beta_0 = 2.5, \gamma_0 = 1.5, p_0 = 1.98$								
n=50	β	2.93	-0.17	00.89	1.04	0.42	0.92	4.95
	p	2.49	-0.26	15.35	3.95	1.99	-3.88	8.88
n=75	β	2.79	-0.12	00.49	0.76	0.30	0.92	4.66
	p	1.93	0.02	02.41	1.55	0.78	-2.25	6.12
n=100	β	2.74	-0.09	00.35	0.64	0.14	1.59	3.88
	p	1.78	0.09	01.43	1.21	0.61	0.14	3.45
n=150	β	2.67	-0.07	00.24	0.52	0.21	1.17	4.17
	p	1.59	0.19	00.47	0.78	0.39	-0.75	3.94
2. $\alpha_0 = 1.25, \beta_0 = 0.5, \gamma_0 = 1.5, p_0 = 2$								
		Mean	Relative bias	variance	RMSE	Scaled RMSE	LL	UL
n=50	β	0.69	-0.39	00.08	0.35	0.72	-00.67	02.07
	p	3.43	-0.72	22.32	4.93	2.46	-11.83	18.83
n=75	β	0.65	-0.30	00.06	0.28	0.57	-00.23	01.53
	p	2.92	-0.46	09.53	3.22	1.61	-03.71	09.55
n=100	β	0.61	-0.21	00.03	0.21	0.41	00.15	01.05
	p	2.49	-0.24	01.95	1.47	0.74	-00.01	04.98
n=150	β	0.58	-0.16	00.02	0.17	0.35	00.23	00.94
	p	2.34	-0.17	00.88	0.99	0.49	00.54	04.13
3. $\alpha_0 = 1.25, \beta_0 = 1.5, \gamma_0 = 1.5, p_0 = 0.5$								
		Mean	Relative bias	variance	RMSE	Scaled RMSE	LL	UL
n=50	β	1.69	-0.13	0.46	0.71	0.47	0.11	3.27
	p	0.84	-0.68	1.05	1.07	2.16	-0.74	2.43
n=75	β	1.64	-0.09	0.25	0.52	0.35	0.68	2.59
	p	0.72	-0.44	0.21	0.51	1.02	0.08	1.35
n=100	β	1.59	-0.06	0.17	0.42	0.28	0.07	3.11
	p	0.65	-0.29	0.09	0.33	0.67	-0.67	1.97
n=150	β	1.55	-0.04	0.11	0.33	0.22	0.75	2.36
	p	0.61	-0.21	0.04	0.22	0.43	0.05	1.16

Table 3. MLE's and mean square error of three parameters (β, γ and p) in the case of the (P-A-L) modified Weibull distribution

1. $\alpha_0 = 1.25, \beta_0 = 2.5, \gamma_0 = 0.5, p_0 = 0.9$						
		Mean	Relative bias	variance	RMSE	Scaled RMSE
n=50	β	2.738	-0.095	0.568	0.791	0.316
	γ	0.501	-0.002	0.005	0.071	0.143
	p	1.285	-0.428	0.784	0.966	1.073
n=100	β	2.670	-0.068	0.312	0.584	0.234
	γ	0.496	0.007	0.003	0.054	0.107
	p	1.216	-0.351	0.457	0.746	0.829
n=150	β	2.627	-0.051	0.167	0.427	0.171
	γ	0.496	0.006	0.002	0.048	0.097
	p	1.142	-0.268	0.320	0.615	0.684
n=200	β	2.627	-0.051	0.167	0.427	0.171
	γ	0.499	0.002	0.002	0.043	0.087
	p	1.120	-0.245	0.306	0.596	0.662
n=250	β	2.587	-0.035	0.149	0.395	0.158
	γ	0.498	0.004	0.001	0.038	0.076
	p	1.088	-0.209	0.268	0.551	0.612
n=300	β	2.578	-0.031	0.133	0.373	0.149
	γ	0.498	0.004	0.001	0.035	0.071
	p	1.076	-0.196	0.245	0.525	0.584
2. $\alpha_0 = 0.2, \beta_0 = 2.2, \gamma_0 = 2.8, p_0 = 0.8$						
		Mean	Relative bias	variance	RMSE	Scaled RMSE
n=50	β	2.529	-0.149	0.305	0.625	0.292
	γ	2.719	0.029	0.211	0.466	0.167
	p	1.461	-0.0826	1.042	1.216	1.520
n=100	β	2.428	-0.104	0.149	0.449	0.204
	γ	2.720	0.028	0.151	0.397	0.142
	p	1.294	-0.618	0.707	0.975	1.219
n=150	β	2.388	-0.085	0.104	0.374	0.169
	γ	2.722	0.028	0.109	0.339	0.121
	p	1.230	-0.538	0.581	0.875	1.094
n=200	β	2.359	-0.072	0.086	0.333	0.151
	γ	2.701	0.035	0.082	0.303	0.108
	p	1.176	-0.470	0.362	0.709	0.887
n=250	β	2.342	-0.065	0.076	0.311	0.141
	γ	2.739	0.022	0.081	0.292	0.104
	p	1.125	-0.406	0.399	0.711	0.888
n=300	β	2.343	-0.065	0.065	0.293	0.133
	γ	2.719	0.029	0.071	0.278	0.099
	p	1.127	-0.409	0.320	0.654	0.817
3. $\alpha_0 = 1.2, \beta_0 = 2, \gamma_0 = 5, p_0 = 6.5$						
		Mean	Relative bias	variance	RMSE	Scaled RMSE
n=50	β	02.353	-0.176	000.428	00.744	0.372
	γ	05.134	-0.026	003.262	01.811	0.362
	p	12.865	-0979	141.060	13.475	2.073
n=100	β	02.237	-0.118	000.267	00.569	0.284
	γ	04.984	0.033	001.414	01.189	0.238
	p	10.503	-0.616	071.433	09.352	1.439
n=150	β	02.158	-0.079	000.195	00.469	0.235
	γ	04.953	0.009	000.903	00.952	0.190
	p	09.652	-0.485	053.834	07.985	1.229
n=200	β	02.141	-0.070	000.167	00.433	0.216
	γ	04.987	0.003	000.691	00.832	0.166
	p	09.101	-0.400	049.919	07.529	1.158
n=250	β	02.084	-0.042	000.126	00.364	0.182
	γ	05.043	-0.008	000.608	00.781	0.156
	p	08.212	-0.263	029.810	05.722	0.880
n=300	β	02.065	-0.033	000.092	00.309	0.155
	γ	05.023	-0.005	000.427	00.653	0.131
	p	07.803	-0.201	018.784	04.525	0.696
4. $\alpha_0 = 1.5, \beta_0 = 3.5, \gamma_0 = 5, p_0 = 7$						
		Mean	Relative bias	variance	RMSE	Scaled RMSE
n=50	β	04.067	-0.162	01.528	1.360	0.388
	γ	04.972	0.005	02.854	1.689	0.338
	p	12.800	-0.828	93.549	11.278	1.611
n=100	β	03.848	-0.099	00.451	0.756	0.216
	γ	04.878	0.024	01.212	1.107	0.222
	p	11.407	-0.629	53.752	8.554	1.222
n=150	β	03.714	-0.061	00.298	0.586	0.167
	γ	04.951	0.009	00.872	0.935	0.187
	p	09.507	-0.358	27.697	5.829	0.833
n=200	β	03.692	-0.055	00.232	0.518	0.148
	γ	04.894	0.021	00.725	0.857	0.172
	p	09.726	-0.389	29.232	6.055	0.865
n=250	β	03.651	-0.043	00.166	0.436	0.125
	γ	04.941	0.012	00.567	0.756	0.151
	p	08.947	-0.278	17.054	4.566	0.652
n=300	β	03.625	-0.036	00.142	0.397	0.113
	γ	04.956	0.008	00.484	0.697	0.139
	p	08.723	-0.246	15.366	4.282	0.612

Table 4. MLE's and mean square error of four parameters (α, β, γ and p) in the case of the (P-A-L) modified Weibull distribution

1. $\alpha_0 = 1.25, \beta_0 = 2.5, \gamma_0 = 0.5, p_0 = 0.9$						
		Mean	Relative bias	variance	RMSE	Scaled RMSE
n=50	α	1.745	-0.396	0001.321	1.252	1.001
	β	2.382	0.047	0001.271	1.133	0.453
	γ	0.488	0.024	0000.009	0.099	0.198
	p	2.654	-1.949	1234.48	35.178	39.087
n=100	α	1.545	-0.236	0000.750	0.915	0.732
	β	2.415	0.033	0000.904	0.954	0.381
	γ	0.485	0.029	0000.004	0.068	0.136
	p	1.217	-0.352	0000.786	0.941	1.046
n=150	α	1.436	-0.149	0000.712	0.864	0.691
	β	2.439	0.024	0000.959	0.981	0.392
	γ	0.491	0.016	0000.003	0.057	0.115
	p	1.179	-0.310	0000.848	0.962	1.069
n=200	α	1.304	-0.043	0004.706	2.170	1.736
	β	2.589	-0.035	0005.201	2.282	0.912
	γ	0.490	0.019	0000.004	0.065	0.131
	p	1.184	-0.315	0000.903	0.992	1.102
n=250	α	1.343	-0.075	0000.489	0.706	0.564
	β	2.478	0.008	0000.747	0.865	0.346
	γ	0.490	0.019	0000.001	0.045	0.090
	p	1.114	-0.237	0000.511	0.746	0.829
n=300	α	1.317	-0.053	0000.480	0.696	0.556
	β	2.495	-0.001	0000.731	0.855	0.342
	γ	0.494	0.010	0000.002	0.043	0.086
	p	1.089	-0.210	0000.462	0.706	0.784
2. $\alpha_0 = 0.2, \beta_0 = 2.2, \gamma_0 = 2.8, p_0 = 0.8$						
		Mean	Relative bias	variance	RMSE	Scaled RMSE
n=50	α	0.389	-0.948	0.251	0.536	2.679
	β	2.304	-0.047	0.352	0.602	0.274
	γ	2.958	-0.056	0.417	0.665	0.237
	p	1.408	-0.760	7.616	2.679	3.533
n=100	α	0.263	-0.316	0.053	0.238	1.193
	β	2.284	-0.038	0.139	0.382	0.174
	γ	2.903	-0.037	0.285	0.543	0.194
	p	0.958	-0.197	0.359	0.533	0.666
		Mean	Relative bias	variance	RMSE	Scaled RMSE
n=150	α	0.232	-0.163	0.018	0.139	0.695
	β	2.257	0.026	0.073	0.276	0.125
	γ	2.831	-0.011	0.094	0.308	0.110
	p	0.940	-0.175	0.152	0.414	0.517
n=200	α	0.219	-0.098	0.013	0.116	0.584
	β	2.255	-0.025	0.054	0.0239	0.108
	γ	2.808	-0.002	0.073	0.271	0.096
	p	0.930	-0.163	0.120	0.370	0.463
n=250	α	0.215	-0.079	0.010	0.101	0.507
	β	2.274	-0.033	0.051	0.237	0.108
	γ	2.789	0.003	0.055	0.236	0.084
	p	0.951	-0.189	0.143	0.407	0.509
3. $\alpha_0 = 1.2, \beta_0 = 2, \gamma_0 = 5, p_0 = 6.5$						
		Mean	Relative bias	variance	RMSE	Scaled RMSE
n=50	α	1.286	-0.072	00.313	0.566	0.472
	β	2.078	-0.039	00.324	0.574	0.287
	γ	5.710	-0.142	04.127	2.152	0.430
	p	8.412	-0.294	30.485	5.843	0.898
n=100	α	1.222	-0.018	00.131	0.362	0.302
	β	2.058	-0.028	00.152	0.394	0.197
	γ	5.263	-0.053	01.237	1.143	0.228
	p	7.522	-0.157	16.419	4.179	0.643
n=150	α	1.195	0.004	00.076	0.275	0.229
	β	2.049	-0.025	00.107	0.330	0.165
	γ	5.150	-0.030	00.706	0.854	0.171
	p	7.178	-0.104	12.110	3.546	0.545
n=200	α	1.190	00.008	0.052	0.229	0.191
	β	2.022	-0.011	00.076	0.277	0.139
	γ	5.125	-0.025	00.484	0.706	0.141
	p	6.943	-0.068	06.926	2.668	0.411
n=250	α	1.212	-0.010	00.043	0.209	0.123
	β	2.023	-0.011	00.060	0.246	0.123
	γ	5.094	-0.018	00.422	0.657	0.131
	p	7.040	-0.083	06.256	2.559	0.393
n=300	α	1.203	-0.002	00.034	0.186	0.155
	β	2.017	-0.008	00.048	0.221	0.110
	γ	5.104	-0.020	00.313	0.569	0.113
	p	6.759	-0.039	04.554	2.149	0.330

4. $\alpha_0 = 1.5, \beta_0 = 3.5, \gamma_0 = 5, p_0 = 7$						
		Mean	Relative bias	variance	RMSE	Scaled RMSE
n=50	α	1.643	-0.095	00.402	0.649	0.433
	β	3.837	-0.096	01.165	1.131	0.323
	γ	5.747	-0.149	03.624	2.045	0.409
	p	9.398	-0.343	29.794	5.962	0.852
n=100	α	1.850	-0.053	00.204	0.459	0.306
	β	3.689	-0.054	00.420	0.675	0.193
	γ	5.277	-0.055	01.177	1.119	0.224
	p	8.740	-0.248	18.488	4.634	0.662
n=150	α	1.561	-0.041	00.126	0.360	0.240
	β	3.656	-0.044	00.263	0.536	0.153
	γ	5.136	-0.027	00.263	0.536	0.153
	p	8.401	-0.200	10.928	3.590	0.513
n=200	α	1.304	-0.043	04.706	2.170	1.736
	β	2.589	-0.035	05.201	2.282	0.912
	γ	0.490	0.019	00.004	0.065	0.131
	p	1.184	-0.315	00.903	0.992	1.102
n=250	α	1.343	-0.075	00.049	0.706	0.564
	β	2.478	0.008	00.747	0.865	0.346
	γ	0.490	0.019	00.001	0.045	0.090
	p	1.114	-0.237	00.511	0.746	0.829
n=300	α	1.317	-0.053	00.480	0.696	0.556
	β	2.495	-0.001	00.731	0.855	0.342
	γ	0.494	0.010	00.002	0.043	0.086
	p	1.089	-0.210	00.462	0.706	0.784
5. $\alpha_0 = 0.8, \beta_0 = 1.5, \gamma_0 = 4.5, p_0 = 2.5$						
		Mean	Relative bias	variance	RMSE	Scaled RMSE
n=50	α	0.933	-0.166	0.212	0.479	0.598
	β	1.519	-0.013	0.196	0.443	0.295
	γ	4.989	-0.108	2.046	1.512	0.335
	p	3.532	-0.412	5.242	2.511	1.004
n=100	α	0.853	-0.067	0.101	0.323	0.403
	β	1.537	-0.025	0.082	0.290	0.194
	γ	4.676	-0.039	0.689	0.848	0.188
	p	3.178	-0.271	3.193	1.912	0.765
n=150	α	0.848	-0.016	0.074	0.277	0.346
	β	1.516	-0.011	0.046	0.215	0.143
	γ	4.621	-0.268	0.407	0.649	0.144
	p	3.035	-0.214	2.122	1.552	0.621
		Mean	Relative bias	variance	RMSE	Scaled RMSE
n=200	α	0.853	-0.066	0.052	0.235	0.294
	β	1.533	-0.023	0.038	0.199	0.133
	γ	4.563	-0.014	0.284	0.536	0.119
	p	3.062	-0.225	1.776	1.446	0.578
n=250	α	0.839	-0.049	0.043	0.212	0.265
	β	1.523	-0.015	0.033	0.183	0.122
	γ	4.540	-0.008	0.218	0.469	0.104
	p	2.949	-0.179	1.396	1.264	0.506
n=300	α	0.835	-0.044	0.039	0.202	0.253
	β	1.527	-0.018	0.025	0.162	0.108
	γ	4.539	-0.008	0.199	0.448	0.099
	p	2.922	-0.168	1.265	1.201	0.480
6. $\alpha_0 = 0.7, \beta_0 = 3.5, \gamma_0 = 4, p_0 = 1.5$						
		Mean	Relative bias	variance	RMSE	Scaled RMSE
n=50	α	0.861	-0.230	0.276	0.549	0.785
	β	3.853	-0.101	1.099	1.106	0.316
	γ	4.391	-0.097	1.759	1.382	0.346
	p	2.213	-0.475	2.171	1.637	1.091
n=100	α	0.765	-0.093	0.105	0.330	0.472
	β	3.649	-0.043	0.344	0.605	0.173
	γ	4.135	-0.034	0.411	0.655	0.164
	p	1.871	-0.247	0.881	1.009	0.673
n=150	α	0.762	-0.089	0.066	0.264	0.378
	β	3.655	-0.044	0.219	0.494	0.141
	γ	4.035	-0.008	0.247	0.498	0.125
	p	1.884	-0.256	0.889	1.018	0.678
n=200	α	0.750	-0.072	0.051	0.232	0.331
	β	3.605	-0.030	0.164	0.419	0.119
	γ	4.054	-0.013	0.187	0.436	0.109
	p	1.816	-0.211	0.617	0.847	0.565
n=250	α	0.743	-0.062	0.042	0.209	0.299
	β	3.587	-0.025	0.144	0.389	0.111
	γ	4.039	-0.009	0.146	0.384	0.096
	p	1.754	-0.169	0.429	0.703	0.469
n=300	α	0.749	-0.071	0.037	0.200	0.286
	β	3.582	-0.024	0.121	0.358	0.102
	γ	3.992	0.002	0.120	0.347	0.087
	p	1.800	-0.201	0.458	0.741	0.494

5 Application for Real Data

These following data from Murthy et al. [8].

Data Set: Failure Times of 50 Components:

0.036, 0.058, 0.061, 0.074, 0.078, 0.086, 0.102, 0.103, 0.114, 0.116, 0.148, 0.183, 0.192, 0.254, 0.262, 0.379, 0.381, 0.538, 0.570, 0.574, 0.590, 0.618, 0.645, 0.961, 1.228, 1.600, 2.006, 2.054, 2.804, 3.058, 3.076, 3.147, 3.625, 3.704, 3.931, 4.073, 4.393, 4.534, 4.893, 6.274, 6.816, 7.896, 7.904, 8.022, 9.337, 10.94, 11.02, 13.88, 14.73, 15.08

We estimate parameters for The (P-A-L) Modified Weibull Distribution by maximum likelihood estimation are given by

$$\hat{\alpha} = 0.433, \quad \hat{\beta} = 8.751 \times 10^{-12}, \quad \hat{\gamma} = 5.336, \quad \hat{p} = 23.425$$

Now, we are studying goodness of fit statistics test by Chi-Square, where the following hypothesis is:

H_0: the data come from The (P-A-L) Modified Weibull Distribution.
H_1: the data does not come from The (P-A-L) Modified Weibull Distribution.

The data were classified in 6 class intervals as shown in Table 5.

Table 5. For goodness of fit test

Class intervals	O_i	E_i	$\frac{(O_i-E_i)^2}{E-i}$
0-	24	6.5022	47.088
1-	2	6.3215	2.9542
2-	3	6.0619	1.5466
3-	6	5.7015	0.0156
4-	4	5.2234	0.2865
5-	0	4.6263	4.6263
6-	2	3.9342	0.9509
7-	2	3.1979	0.4488
8-	1	2.4827	0.8855
9-	1	1.8466	0.3881
10-	1	1.3239	0.0793
11-	1	0.9218	0.0066
12-	0	0.6278	0.6278
13-	1	0.4209	0.7968
14-	1	0.2791	1.8623
15-16	1	0.1837	3.6285
Total			66.1919

The calculated statistic value of Chi-square test (66.1919) is less than the tabulated statistic value (66.3386). Therefore the null hypothesis do not reject that the data came from The (P-A-L) Modified Weibull Distribution. We use software Mathematica 10 to get the goodness of fit test.

6 Conclusions

In this paper new lifetime distribution is presented which called The (P-A-L) Modified Weibull Distribution (PMWD). The hazard shape for The (P-A-L) Modified Weibull Distribution has different types of shapes such as decreasing and increasing hazard function. For example, if $p \to 1$, β=1 and γ=1 then h(x)=α+1 is constant, whereas if $p \to 1$ and β=1, then $h(x) = \alpha + \gamma x^{\gamma-1}$, which is increasing for $\gamma > 1$ and decreasing for $\gamma < 1$. The maximum likelihood estimates of the parameters and their variance covariance matrix were derived using software Mathematica 10 for numerically solving equations have no analytical solutions. By results of the simulation study, show that the bias for any estimator is decreased when the sample size increases also the relative bias is decreased when the sample size increases. Mean square error (MSE), root mean square error, relative mean square error decrease when the sample size increases. By application on real data, we found that the $PMWD(\alpha, \beta, \gamma, p)$ fits the data.

Acknowledgement

I would like to thank here a number of persons who contributed directly or indirectly to the production of this paper and Special thanks for Prof. Mahmoud Riad Mahmoud, Professor of Mathematical Statistics and Dr. Rasha Mohamed Mandouh, Lecturer of Mathematical Statistics.

Competing Interests

Authors have declared that no competing interests exist.

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