Journal of Advances in Mathematics and Computer Science

26(2): 1-7, 2018; Article no.JAMCS.38654

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Relations between Various Magnitudes Which Serve as Signs of a Right-angled Triangle

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2018/38654
Editor(s):
(1) Huchang Liao, Professor, Business School, Sichuan University, P. R. China.
Reviewers:
(1) Halil Görgün, Dicle University, Turkey.
(2) Alok Kumar Pandey, Roorkee Institute of Technology, India.
(3) Emanuel Guariglia, University of Salerno, Italy.
Complete Peer review History: http://www.sciencedomain.org/review-history/22883

Received: 5th October 2017
Accepted: 6th January 2018
Published: 27th January 2018

Original Research Article

Abstract

We present a proven scheme of 15 relations between the sides of a triangle, the radii of incircles and excircles of the triangle, the area and half the perimeter of the triangle, which serve as signs of a right-angled triangle, together with the 16th relation which is the **Pythagorean Theorem**. The relations are investigated dynamically using the computerized software.

Keywords: Relations between magnitudes in a right-angled triangle; dynamic investigation in a right-angled triangle.

1 Introduction

Relations between different magnitudes in a triangle, such as the radius of the circumcircle of the triangle, the radius of the incircle of the triangle, the radii of the excircles the triangle (which are tangent to one side from the outside and the continuations of the two other sides), as well as their relations with the side lengths the triangle and certain segments in the triangle, are a fascinating subject that can be investigated by

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hypothesizing and investigating the hypotheses dynamically using a computerized technological tool, or using formal mathematical proofs employing various mathematical tools and algebraic manipulations.

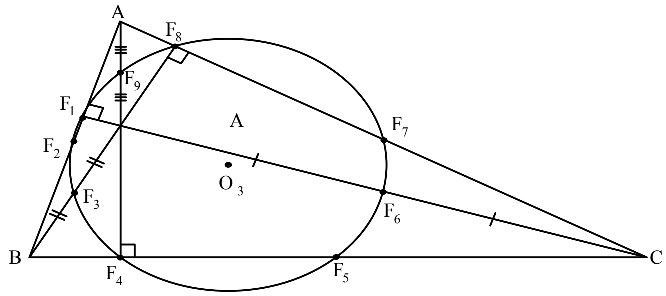


Fig. 1. The 9 point circle

In connection with the excircles, the incircle, and the circumcircle, one should mention **the nine-point circle** [1-2] discovered by **Leonard Euler** in 1765. The nine points in the triangle through which the circle passes:

- The middles of the sides;
- The feet of the altitudes – the points at which and attitudes meet the sides;
- The middles of the segments that connect the vertices of the triangle and the point of intersection of the altitudes.

The nine point circle for an arbitrary triangle ΔABC is shown in Fig. 1.

Later, in 1822, Karl Wilhelm Feuerbach showed that this circle is also tangent to the excircles of the triangle [2-3]. This led to increased interest in finding the relations between the radii and various segments.

It is important to note that the center of the nine-point circle lies on "Euler's line", which passes through the point of intersection of the medians, the point of intersection of the altitudes and the point of intersection of the mid-perpendiculars. The length of the radius of the nine-point circle is equal to half the length of the radius of the circumcircle of the triangle ΔABC.

Several relations are known between the radii, one of which is $\frac{1}{r} = \frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c}$ [4].

Although the geometric properties and some of the connections have been known for many years, the issue is still relevant and appears in some publications [5-8]. Interesting connections exist in a right-angled triangle in the context of Pythagorean Theorem and its various representations.

The current activities are focused on finding 15 relations between various magnitudes that constitute signs for the existence of a right-angled triangle which supplement the well-known relation of the Pythagorean Theorem. Any relation is a necessary and sufficient condition for the triangle to be right-angled. Each relation is accompanied by a mathematical proof based on several known relations between the magnitudes in the triangle and its excircles, incircle and well-known geometrical properties.

2 Development of the Relations that Constitute Signs for a Right-angled Triangle

Notation:

$p = \frac{a+b+c}{2}$ is half the perimeter of the triangle ΔABC

r is the radius of the incircle of the triangle ΔABC

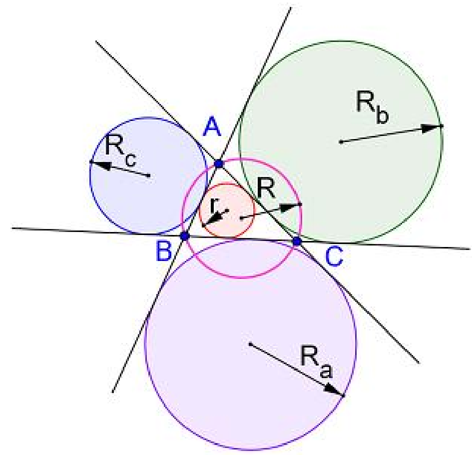


Fig. 2. The triangle and its incircle, excircles and circumcircle

R_a is the radius of the excircle of the triangle and is tangent to the side a.

R_b is the radius of the excircle of the triangle and is tangent to the side b.

R_c is the radius of the excircle of the triangle and is tangent to the side c.

R is the radius of the circumcircle of the triangle ΔABC
The triangle and the radii of the different circles are shown in Fig. 2.

A GeoGebra applet was prepared which allows the vertices of the triangle to be dragged, and thus the lengths of its sides changed, resulting in an immediate demonstration of the effect on the lengths of the radii, which appear on screen at every stage.

It is possible to reach the applet using the following link:

Link 1: https://www.geogebra.org/m/TueeM84U

Known relations:

$$r = \frac{S_\Delta}{p}$$

$$R_a = \frac{S_\Delta}{p-a}$$

$$R_b = \frac{S_\Delta}{p-b}$$

$$R_c = \frac{S_\Delta}{p-c}$$

$$a+b-c = 2(p-c)\ ,\ a+c-b = 2(p-b)\ ,\ b+c-a = 2(p-a)$$

The right-angled triangle and its circles, together with the scheme of the relations between the magnitudes that constitute a precondition for the formation of such a triangle. In Fig. 3.

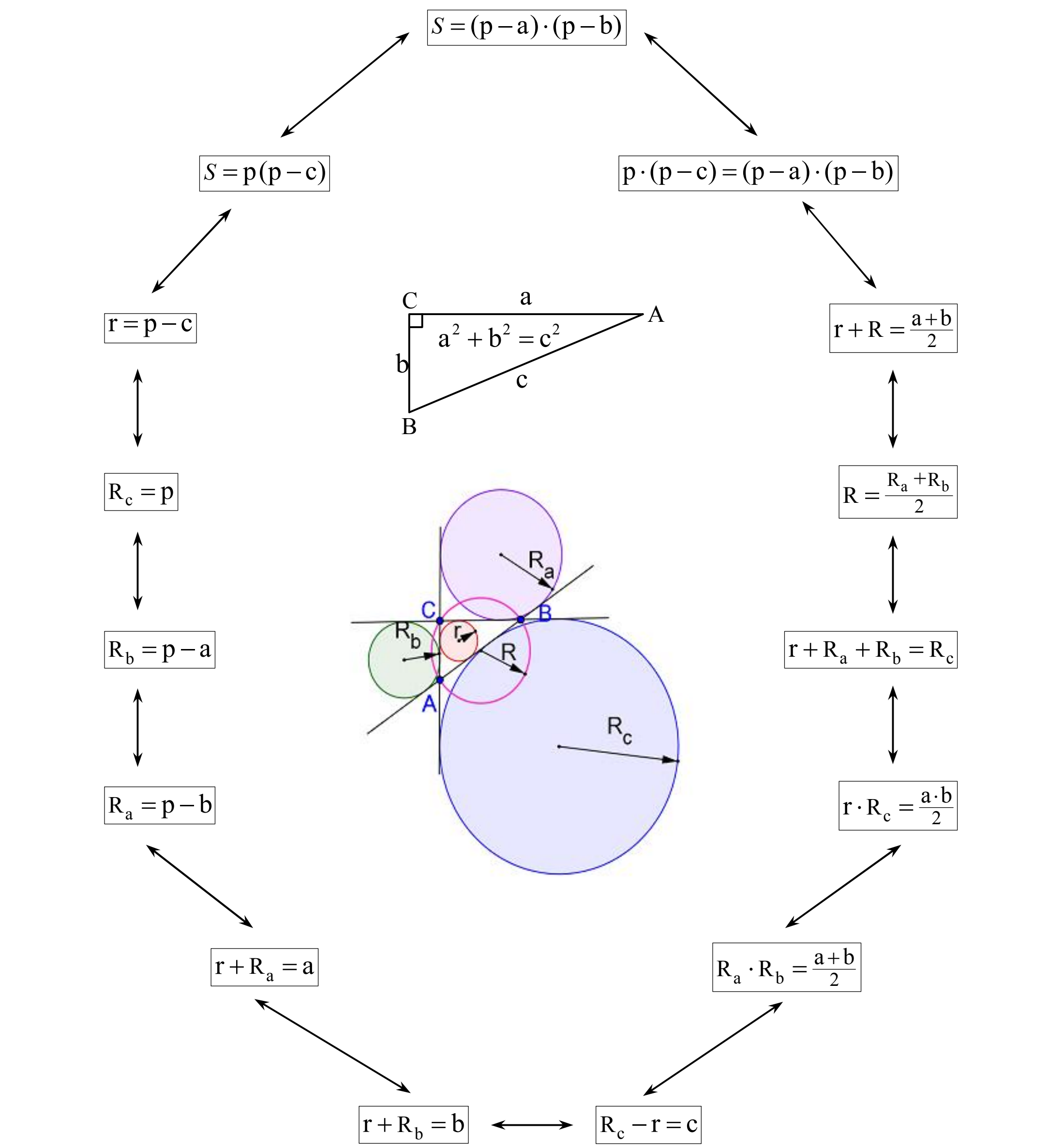


Fig. 3. A right-angled triangle, its circles and the scheme of the relations between its magnitudes

3 Mathematical Development of the Relations between the Magnitudes in a Right-angled Triangle

(1) $S=\frac{a\cdot b}{2} \;\Rightarrow\; a\cdot b=2S$

(2) $a^2+b^2=c^2 \;\Rightarrow\; a^2+b^2-c^2=0 \;\Rightarrow\; a^2+b^2-c^2+2a\cdot b=2a\cdot b=4S$

$\Downarrow$

(3) $(a+b)^2 - c^2 = 4S \quad \Rightarrow \qquad (a+b+c)\cdot(a+b-c) = 4p\cdot(p-c) = 4S$

(4) $(a+b)^2 - c^2 = 4S \quad \Rightarrow \qquad S = p\Downarrow(p-c)$

(5) $S = \sqrt{p(p-a)(p-b)(p-c)} = p(p-c) \quad \Rightarrow \quad (p-a)\cdot(p-b) = p(p-c) = S$

(6) $r = \frac{S}{p} = \frac{p(p-c)}{p} = p-c$

(7) $R_a = \frac{S}{p-a} = \frac{(p-a)\cdot(p-b)}{p-a} = p-b$

(8) $R_b = \frac{S}{p-b} = \frac{(p-a)\cdot(p-b)}{p-b} = p-a$

(9) $R_C = \frac{S}{p-c} = \frac{p\cdot(p-c)}{p-c} = p$

(10) $R_a + R_b + R_c = p-b+p-a = c = 2R \quad \Rightarrow \quad R = \frac{R_a + R_b}{2}$

(11) $R + r = \frac{a+b}{2}$

looking at Fig. 4, one can write down the following relations:

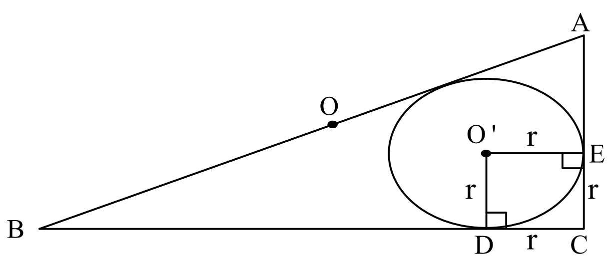


Fig. 4. The relation between the radius of the incircle, the radius of the circumcircle and the legs of the triangle

$BF = BD = a - r$

$AE = AF = b - r$

$AB = c = 2R = BF + AF = a - r + b - r \quad \Rightarrow \quad 2R + 2r = a + b$

$$\Downarrow$$

$$R + r = \frac{a+b}{2}$$

(12) $r + R_a = p - c + p - b = a \qquad (6\,,7)$

(13) $r + R_b = p - c + p - a = b \qquad (6\,,8)$

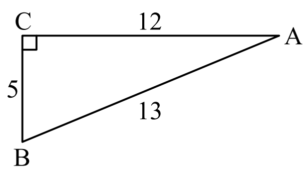
(14) $R_c - r = p - (p - c) = c \qquad (6\,,9)$

(15) $r + R_a + R_b = p - c + p - b + p - a = p = R_c \qquad (6\,,7\,,8)$

(16) $r \cdot R_c = (p-c)\cdot p = \frac{a\cdot b}{2} \qquad (3\,,6\,,9)$

$$(17)\ R_a \cdot R_b = (p-b)\cdot(p-a) = \frac{a\cdot b}{2} \qquad (7\ ,8)$$

4 Numerical Example



$$S = 30\ ,\ p = 15\ ,\ r = 2\ ,\ R_a = 3\ ,\ R_b = 10\ ,\ R_c = 15\ ,\ p-a = 10\ ,\ p-b = 3\ ,\ p-c = 2$$

$$(p-a)\cdot(p-b) = 3\cdot 10 = 30$$

$$p\cdot(p-c) = 15\cdot 2 = 30$$

$$R = \frac{c}{2} = 6.5\ ,\ R = \frac{R_a + R_b}{2} = 6.5$$

A GeoGebra applet was prepared in order to illustrate the formulas, which contains a right-angled triangle with a fixed vertex C and vertices A and B that can be dragged, changing the lengths of the sides and accordingly – the radii of the circles shown on the screen. In each stage, the screen shows the values of the radii, the side lengths of the triangle, the area, and half the perimeter.
One can reach the applet using Link 2:

Link 2: https://www.geogebra.org/m/p3k9KJzg

Note: the proof of the relations between various magnitudes that constitute signs for the existence of a right-angled triangle was based on the fact that the triangle is given. However, in order to prove that these relations are necessary and sufficient conditions one must also provide the proof in the other direction, where the relation is given and one has to prove that it necessarily gives a right-angled triangle. In some of the relations the proof is trivial and in others, a mathematical effort is required in order to obtain the proof. A check was made that this is true, and the reader should confirm this.

5 Summary

We presented a scheme of relations between the side lengths of a triangle, the radii of its excircles and incircle, the circumcircle and half the perimeter of the triangle. When one of the relations holds in itself, this means that the triangle is right-angled.

Competing Interests

Authors have declared that no competing interests exist.

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Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sciencedomain.org/review-history/22883