Journal of Advances in Mathematics and Computer Science

26(2): 1-16, 2018; Article no.JAMCS.38768

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Explicit Determinants and Inverses of Skew Circulant and Skew Left Circulant Matrices with the Pell-Lucas Numbers

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Authors' contributions

This work was carried out in collaboration between all authors. Author JY designed the study, performed the statistical analysis and wrote the first draft of the manuscript. Author JS managed the analyses of the study. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2018/38768
Editor(s):
(1) Dariusz Jacek Jakobczak, Assistant Professor, Chair of Computer Science and Management in this Department, Technical University of Koszalin, Poland.
Reviewers:
(1) Omur Deveci, Kafkas University, Turkey.
(2) Grienggrai Rajchakit, Maejo University, Thailand.
(3) Ahmet Oteles, Dicle University, Turkey.
(4) Mustafa Bahsi, Aksaray University, Turkey.
Complete Peer review History: http://www.sciencedomain.org/review-history/22872

Received: ***12th November 2017***
Accepted: ***17th January 2018***
Published: ***26th January 2018***

Original Research Article

Abstract

In this paper, we consider the skew circulant and skew left circulant matrices with the Pell-Lucas numbers. We discuss the invertibility of the skew circulant and skew left circulant matrices and present the determinant and the inverse matrix of them by constructing the transformation matrices.

Keywords: Skew circulant matrix; skew left circulant matrix; determinant; inverse; Pell-Lucas number.

2010 Mathematics Subject Classification: 15A09, 15A15, 15B05, 11B39.

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1 Introduction

Skew circulant and circulant matrices have important applications in various disciplines [1], [2], [3], such as image processing, communications, signal processing, encoding, preconditioner, and solving least squares problems. In [4], [5], [6],[7], the skew-circulant matrix as pre-conditioners for LMF-based ODE codes, Hermitian and skew-Hermitian Toeplitz systems were considered. Lyness [8] constructed an s-dimensional lattice rules by a skew-circulant matrix. In [9], the authors discussed the spectral decompositions of skew circulant and skew left circulant matrices.

Recently, there are several papers on the Pell sequences and circulant matrices. M. Akbulak et al. [10] gave sum formulas of Pell and Jacobsthal sequences by matrix method. A. Öteles et al. investigated permanents of an $n \times n$ (0,1,2)-matrix by contraction method and showed that the permanent of the matrix is equal to the generalized k-Pell numbers in [11]. In [12], [13], [14], the authors defined the recurrence sequences by using the circulant matrices which are obtained from the characteristic polynomial.

The Pell-Lucas sequences are defined by the following recurrence relations, respectively [15, 16, 17, 18]:

$$Q_{n+1} = 2Q_n + Q_{n-1}, \qquad \text{where } Q_0 = 2, \quad Q_1 = 2, \text{ for } n \geq 0,$$

the first few values of the sequences are given by the following table:

n	0	1	2	3	4	5	6	7	8	9
Q_n	2	2	6	14	34	82	198	478	1154	2786

The $\{Q_n\}$ is given by the formula $Q_n = \alpha^n + \beta^n$, where α and β are the roots of the characteristic equation $x^2 - 2x - 1 = 0$.

The purpose of this paper is to obtain the determinants and inverses of skew circulant and skew left circulant matrices by using some properties of Pell and Pell-Lucas numbers . In the following, let r be a nonegative integer. We adopt the following two conventions $0^0 = 1$, and for any sequence $\{a_n\}$, $\sum_{k=i}^{n} a_k = 0$ if $i > n$.

Definition 1.1 ([9])**.** A skew circulant matrix with the first row $(a_1, a_2, \ldots, a_n)$ is a square matrix of the form

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_{n-1} & a_n \\ -a_n & a_1 & a_2 & & a_{n-1} \\ \vdots & -a_n & a_1 & \ddots & \vdots \\ -a_3 & & \ddots & \ddots & a_2 \\ -a_2 & -a_3 & \ldots & -a_n & a_1 \end{pmatrix}_{n\times n}$$

denoted by $\mathrm{SCirc}(a_1, a_2, ..., a_n)$.

Definition 1.2 ([9])**.** A skew left circulant matrix with the first row $(a_1, a_2, \ldots, a_n)$ is a square matrix of the form

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_{n-1} & a_n \\ a_2 & & a_{n-1} & a_n & -a_1 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ a_{n-1} & a_n & -a_1 & & -a_{n-2} \\ a_n & -a_1 & \ldots & -a_{n-2} & -a_{n-1} \end{pmatrix}_{n\times n}$$

denoted by $\mathrm{SLCirc}(a_1, a_2, ..., a_n)$.

Lemma 1.1 ([1], [9]). *Let* $A := \mathrm{SCirc}(a_1, a_2, ..., a_n)$. *Then we have*
(i) A *is invertible if and only if* $f(\omega^k \eta) \neq 0$ $(k = 0, 1, 2, ..., n-1)$, *where* $f(x) = \sum_{j=1}^{n} a_j x^{j-1}$, $\omega = \exp(\frac{2\pi \mathrm{i}}{n})$ *and* $\eta = \exp(\frac{\pi \mathrm{i}}{n})$;
(ii) *if* A *is invertible, then the inverse of* A *is a skew circulant matrix.*

Lemma 1.2 ([19]). *Let* $A := \mathrm{SLCirc}(a_1, a_2, ..., a_n)$ *and* n *be odd. Then*

$$\lambda_j = \pm |\sum_{k=1}^{n} a_k \omega^{(j-\frac{1}{2})(k-1)}|, \ (j = 1, 2, \ldots, \frac{n-1}{2}),$$
$$\lambda_{\frac{n+1}{2}} = \sum_{k=1}^{n} |a_k (-1)^{k-1}|,$$

where λ_j $(j = 1, 2, \ldots, \frac{n-1}{2}, \frac{n+1}{2})$ *are the eigenvalues of* A.

Lemma 1.3. *For the orthogonal skew left circulant matrix*

$$\Theta := \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \vdots & & \iddots & -1 \\ \vdots & \vdots & \iddots & -1 & 0 \\ \vdots & 0 & \iddots & \iddots & \vdots \\ 0 & -1 & 0 & \cdots & 0 \end{pmatrix}_{n \times n},$$

we have

$$\mathrm{SCirc}(a_1, a_2, \ldots, a_n) = \Theta \mathrm{SLCirc}(a_1, a_2, \ldots, a_n).$$

Lemma 1.4. *If* $[\mathrm{SCirc}(a_1, a_2, ..., a_n)]^{-1} = \mathrm{SCirc}(b_1, b_2, ..., b_n)$, *then*

$$[\mathrm{SLCirc}(a_1, a_2, ..., a_n)]^{-1} = \mathrm{SLCirc}(b_1, -b_n, ..., -b_2).$$

Proof. Let $B := \mathrm{SCirc}(a_1, a_2, ..., a_n)$ and $A' := \mathrm{SLCirc}(a_1, a_2, ..., a_n)$. By Lemma 1.3, we have $B = \Theta A'$. Thus, we obtain $A'^{-1} = B^{-1}\Theta = \mathrm{SLCirc}(b_1, -b_n, ..., -b_2)$, where $B^{-1} = \mathrm{SCirc}(b_1, b_2, ..., b_n)$. □

Lemma 1.5 ([20]). *Let* $\{Q_n\}$ *be Pell-Lucas numbers. Then*

$$\text{(i)} \sum_{i=0}^{n-1} Q_i = \frac{1}{2}(Q_n + Q_{n-1}), \tag{1.1}$$

$$\text{(ii)} \sum_{i=0}^{n-1} Q_i^2 = \frac{1}{2} Q_n Q_{n-1} + 2, \tag{1.2}$$

$$\text{(iii)} \sum_{i=0}^{n-1} i Q_i = \frac{1}{2}[(n-2)Q_n + (n-1)Q_{n-1} + 2]. \tag{1.3}$$

Proof. Since $Q_0 = 2$, $Q_1 = 2$ and by [20], we have (i) and (ii).

(iii) By Equation (1.1),

$$\sum_{i=1}^{n-1} iQ_i = \sum_{k=1}^{n-1}\sum_{i=n-k}^{n-1} Q_i = \sum_{k=1}^{n-1}(\sum_{i=0}^{n-1} Q_i - \sum_{i=0}^{n-k-1} Q_i)$$
$$= \frac{1}{2}\sum_{k=1}^{n-1}(Q_n + Q_{n-1} - Q_{n-k} - Q_{n-k-1})$$
$$= \frac{1}{2}[(n-2)Q_n + (n-1)Q_{n-1} + 2].$$

The proof is completed. □

Definition 1.3 ([21])**.** Let $A = (a_{ij})$ be an $n \times n$ matrix. The Euclidean (or Frobenius) norm, the spectral norm, the maximum column sum matrix norm and the maximum row sum matrix norm of the matrix A are, respectively:

$$\|A\|_F = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}, \tag{1.4}$$

$$\|A\|_2 = \left(\max_{1\le i\le n} \lambda_i(A^*A)\right)^{\frac{1}{2}}, \tag{1.5}$$

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|, \tag{1.6}$$

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|, \tag{1.7}$$

where A^* denotes the conjugate transpose of A.

Lemma 1.6 ([22])**.** *If A is an $n \times n$ real symmetric or normal matrix, then we have*

$$\|A\|_2 = \max_{1\le i\le n} |\lambda_i|, \tag{1.8}$$

where λ_i $(i = 1, 2, \ldots, n)$ are the eigenvalues of A.

Definition 1.4 ([23])**.** Let $A = (a_{ij})$ be an $n \times n$ matrix with eigenvalues λ_i $(i = 1, 2, ...n)$. The spread of A is defined as

$$s(A) = \max_{i,j} \mid \lambda_i - \lambda_j \mid . \tag{1.9}$$

In [23], authors have obtained an upper bound for the spread of a matrix, which states that

$$s(A) \le \sqrt{2 \parallel A \parallel_F^2 - \frac{2}{n} \mid \mathrm{tr}A \mid^2}, \tag{1.10}$$

where $\parallel A \parallel_F$ denotes the Frobenius norm of A and tr$A$ is trace of A.

Lemma 1.7 ([24])**.** *Let $A = (a_{ij})$ be an $n \times n$ matrix. Then*
(i) *if A is real and normal, then*

$$s(A) \ge \frac{1}{n-1}|\sum_{i\ne j} a_{ij}|; \tag{1.11}$$

(ii) *if A is Hermitian, then*

$$s(A) \ge 2\max_{i\ne j} |a_{ij}|. \tag{1.12}$$

2 Determinant and Inverse of Skew Circulant Matrix with the Pell-Lucas Numbers

In this section, let $D_{r,n} := \text{SCirc}(Q_{r+1}, Q_{r+2}, \ldots, Q_{r+n})$. Firstly, we give a determinant explicit formula for the matrix $D_{r,n}$. Afterwards, we prove that $D_{r,n}$ is an invertible matrix for any positive integer n, and then we find the inverse of the matrix $D_{r,n}$.

Theorem 2.1. *Let* $D_{r,n} := \text{SCirc}(Q_{r+1}, Q_{r+2}, \ldots, Q_{r+n})$*. Then we have*

$$\det D_{r,n} = Q_{r+1} \cdot \left[Q_{r+1} + \theta Q_{r+n} + \sum_{k=1}^{n-2} (\theta Q_{r+k+1} - Q_{r+k+2}) \cdot y^{n-(k+1)} \right] \cdot (Q_{r+1} + Q_{r+n+1})^{n-2}, \tag{2.1}$$

where Q_{r+n} *is the* $(r+n)$*th Pell-Lucas number,* $\theta = \frac{Q_{r+2}}{Q_{r+1}}$ *and* $y = -\frac{Q_r + Q_{r+n}}{Q_{r+1} + Q_{r+n+1}}$.

Proof. Obviously, $\det D_{r,2} = Q_{r+1}^2 + Q_{r+2}^2$ satisfies the equation (1.1). When $n > 2$, we introduce two $n \times n$ matrices as follows:

$$\Sigma = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \theta & \vdots & & & & & \iddots & 1 \\ 1 & \vdots & & & & \iddots & 1 & -2 \\ 0 & \vdots & & & \iddots & 1 & -2 & -1 \\ \vdots & \vdots & & \iddots & \iddots & \iddots & \iddots & 0 \\ \vdots & \vdots & \iddots & 1 & \iddots & \iddots & \iddots & \vdots \\ \vdots & 0 & 1 & -2 & \iddots & \iddots & & \vdots \\ 0 & 1 & -2 & -1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \text{ and } \Omega_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & (y)^{n-2} & \vdots & & \iddots & 1 \\ \vdots & (y)^{n-3} & \vdots & \iddots & 1 & 0 \\ \vdots & \vdots & 0 & \iddots & \iddots & \vdots \\ \vdots & y & 1 & \iddots & & \vdots \\ 0 & 1 & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

where $y = -\dfrac{Q_r + Q_{r+n}}{Q_{r+1} + Q_{r+n+1}}$.

And we have

$$\Sigma D_{r,n} \Omega_1 = \begin{pmatrix} Q_{r+1} & q_n' & Q_{r+n-1} & \cdots & \cdots & Q_{r+3} & Q_{r+2} \\ 0 & q_n & \theta Q_{r+n-1} - Q_{r+n} & \cdots & \cdots & \theta Q_{r+3} - Q_{r+4} & \theta Q_{r+2} - Q_{r+3} \\ \vdots & 0 & Q_{r+1} + Q_{r+n+1} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & Q_r + Q_{r+n} & \ddots & \ddots & & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & Q_{r+1} + Q_{r+n+1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & Q_r + Q_{r+n} & Q_{r+1} + Q_{r+n+1} \end{pmatrix},$$

where

$$\theta = \frac{Q_{r+2}}{Q_{r+1}}, \; q_n = Q_{r+1} + \theta Q_{r+n} + \sum_{k=1}^{n-2} (\theta Q_{r+1+k} - Q_{r+2+k}) \cdot y^{n-(k+1)}, \; q_n' = \sum_{k=1}^{n-1} Q_{r+k+1} \cdot y^{n-(k+1)}.$$

Hence, we obtain

$$\begin{aligned}\det\Sigma\det D_{r,n}\det\Omega_1 &= Q_{r+1}q_n\cdot(Q_{r+1}+Q_{r+n+1})^{n-2}\\ &= Q_{r+1}\cdot\left[Q_{r+1}+\theta Q_{r+n}+\sum_{k=1}^{n-2}(\theta Q_{r+k+1}-Q_{r+k+2})\cdot y^{n-(k+1)}\right]\cdot(Q_{r+1}+Q_{r+n+1})^{n-2}.\end{aligned}$$

While

$$\det\Sigma=\det\Omega_1=(-1)^{\frac{(n-1)(n-2)}{2}},$$

we have

$$\det D_{r,n}=Q_{r+1}\cdot\left[Q_{r+1}+\theta Q_{r+n}+\sum_{k=1}^{n-2}(\theta Q_{r+k+1}-Q_{r+k+2})\cdot y^{n-(k+1)}\right]\cdot(Q_{r+1}+Q_{r+n+1})^{n-2}.$$

□

Theorem 2.2. *Let $D_{r,n}:=\mathrm{SCirc}(Q_{r+1},Q_{r+2},\dots,Q_{r+n})$. Then $D_{r,n}$ is invertible for any positive integer n .*

Proof. Since $Q_{r+n}=\alpha^{r+n}+\beta^{r+n}$, where $\alpha+\beta=2,\ \alpha\cdot\beta=-1$ and $\eta=\exp(\frac{\pi\mathrm{i}}{n})$, we have

$$\begin{aligned}f(\omega^k\eta) &= \sum_{j=1}^{n}Q_{r+j}(\omega^k\eta)^{j-1}=\sum_{j=1}^{n}(\alpha^{r+j}+\beta^{r+j})(\omega^k\eta)^{j-1}\\ &= \frac{\alpha^r(1+\alpha^n)}{1-\alpha\omega^k\eta}+\frac{\beta^r(1+\beta^n)}{1-\beta\omega^k\eta}\ \text{(because } 1-\alpha\omega^k\eta\neq 0,\ 1-\beta\omega^k\eta\neq 0)\\ &= \frac{(\alpha^{r+1}+\beta^{r+1})+(\alpha^{r+n+1}+\beta^{r+n+1})-\alpha\beta(\alpha^{r+n}+\beta^{r+n})\omega^k\eta-\alpha\beta(\alpha^r+\beta^r)\omega^k\eta}{1-(\alpha+\beta)\omega^k\eta+\alpha\beta\omega^{2k}\eta^2}\\ &= \frac{Q_{r+1}+Q_{r+n+1}+(Q_r+Q_{r+n})\omega^k\eta}{1-2\omega^k\eta-\omega^{2k}\eta^2},\quad (k=1,2,...,n-1).\end{aligned}$$

If there exists $\omega^l\eta(l=1,2,\dots,n-1)$ such that $f(\omega^l\eta)=0$, then we obtain $Q_{r+1}+Q_{r+n+1}+(Q_r+Q_{r+n})\omega^l\eta=0$ for $1-2\omega^l\eta-\omega^{2l}\eta^2\neq 0$, thus, $\omega^l\eta=-\frac{Q_{r+1}+Q_{r+n+1}}{Q_r+Q_{r+n}}$ is a real number, while

$$\omega^l\eta=\exp\frac{(2l+1)\pi\mathrm{i}}{n}=\cos\frac{(2l+1)\pi}{n}+\mathrm{i}\sin\frac{(2l+1)\pi}{n}.$$

Hence, $\sin\frac{(2l+1)\pi}{n}=0$, and we have $\omega^l\eta=-1$ for $0<\frac{(2l+1)\pi}{n}<2\pi$. But $x=-1$ is not the root of the equation $Q_{r+1}+Q_{r+n+1}+(Q_r+Q_{r+n})x=0$ for any positive integer n. Hence, we obtain $f(\omega^k\eta)\neq 0$ for any $\omega^k\eta(k=1,2,\dots,n-1)$, while

$$f(\eta)=\sum_{j=1}^{n}Q_j\eta^{j-1}=\frac{Q_{r+1}+Q_{r+n+1}+(Q_r+Q_{r+n})\eta}{1-2\eta-\eta^2}\neq 0.$$

Thus, by Lemma 1.1, the proof is completed. □

Lemma 2.3. *Let the matrix $\mathcal{H}=[h_{ij}]_{i,j=1}^{n-2}$ be of the form*

$$h_{ij}=\begin{cases}Q_{r+1}+Q_{r+n+1}, & i=j,\\ Q_r+Q_{r+n}, & i=j+1,\\ 0, & \text{otherwise}.\end{cases}$$

Then $\mathcal{H}^{-1}=[h'_{ij}]_{i,j=1}^{n-2}$ is

$$h'_{ij}=\begin{cases}\frac{(-Q_r-Q_{r+n})^{i-j}}{(Q_{r+1}+Q_{r+n+1})^{i-j+1}}, & i\geq j,\\ 0, & i<j.\end{cases}$$

Proof. Let $r_{ij} = \sum_{k=1}^{n-2} h_{ik}h'_{kj}$. Obviously, $r_{ij} = 0$ for $i < j$. For $i = j$, we obtain

$$r_{ii} = h_{ii}h'_{ii} = (Q_{r+1} + Q_{r+n+1}) \cdot \frac{1}{Q_{r+1} + Q_{r+n+1}} = 1.$$

For $i \geq j + 1$, we obtain

$$\begin{aligned} r_{ij} &= \sum_{k=1}^{n-2} h_{ik}h'_{kj} = h_{i(i-1)}h'_{(i-1)j} + h_{ii}h'_{ij} \\ &= (Q_r + Q_{r+n}) \cdot \frac{(-Q_r - Q_{r+n})^{i-j-1}}{(Q_{r+1} + Q_{r+n+1})^{i-j}} + (Q_{r+1} + Q_{r+n+1}) \cdot \frac{(-Q_r - Q_{r+n})^{i-j}}{(Q_{r+1} + Q_{r+n+1})^{i-j+1}} = 0. \end{aligned}$$

Hence, we verify $\mathcal{H}\mathcal{H}^{-1} = I_{n-2}$, where I_{n-2} is $(n-2) \times (n-2)$ identity matrix. Similarly, we can verify $\mathcal{H}^{-1}\mathcal{H} = I_{n-2}$. Thus, the proof is completed. □

Theorem 2.4. *Let* $D_{r,n} := \mathrm{SCirc}(Q_{r+1}, Q_{r+2}, \ldots, Q_{r+n})$. *Then we have*

$$D_{r,n}^{-1} = \frac{1}{q_n} SCirc(y'_1, y'_2, \ldots, y'_n),$$

where

$$\begin{aligned} y'_1 &= 1 - \sum_{i=1}^{n-3} (Q_{r+n+2-i} - \theta Q_{r+n+1-i}) \cdot \frac{(-d)^{i-1}}{c^i} - 2(Q_{r+3} - \theta Q_{r+2}) \frac{(-d)^{n-3}}{c^{n-2}}, \\ y'_2 &= -\theta - \sum_{i=1}^{n-2} (Q_{r+n+1-i} - Q_{r+n-i}) \frac{(-d)^{i-1}}{c^i}, \\ y'_3 &= -(Q_{r+3} - \theta Q_{r+2}) \cdot \frac{1}{c}, \\ y'_4 &= -\sum_{i=1}^{2} (Q_{r+1+i} - \theta Q_{r+i}) \frac{(-d)^{i-1}}{c^i}, \\ &\cdots \\ y'_k &= -\sum_{i=1}^{2} (Q_{r+1+i} - \theta Q_{r+i}) \cdot \frac{(-d)^{k-5+i}}{(c)^{k-4+i}}, (k = 5, 6, \ldots, n), \\ \theta &= \frac{Q_{r+2}}{Q_{r+1}}, c = Q_{r+n+1} + Q_{r+1}, \ d = Q_{r+n} + Q_r. \end{aligned}$$

Proof. Let

$$\Omega_2 = \begin{pmatrix} 1 & -\frac{q'_n}{Q_{r+1}} & \mu_{1,3} & \mu_{1,4} & \cdots & \mu_{1,n} \\ 0 & 1 & \mu_{2,3} & \mu_{2,4} & \cdots & \mu_{2,n} \\ \vdots & \ddots & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & 1 & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix},$$

where

$$\mu_{1,3} = \frac{1}{Q_{r+1}}[\frac{q_n'}{q_n}(\theta Q_{r+n-1} - Q_{r+n}) - Q_{r+n-1}],$$
$$\mu_{1,k} = \frac{1}{Q_{r+1}}[\frac{q_n'}{q_n}(\theta Q_{r+n-(k-2)} - Q_{r+n-(k-3)}) - Q_{r+n-(k-2)}], (k = 4, 5, \ldots, n),$$
$$\mu_{2,3} = \frac{Q_{r+n} - \theta Q_{r+n-1}}{q_n},$$
$$\mu_{2,k} = \frac{Q_{r+n+3-k} - \theta Q_{r+n+2-k}}{q_n}, (k = 4, 5, \ldots, n),$$
$$\theta = \frac{Q_{r+2}}{Q_{r+1}}, \quad q_n' = \sum_{k=1}^{n-1} Q_{r+k+1} \cdot y^{n-(k+1)},$$
$$q_n = Q_{r+1} + \theta Q_{r+n} + \sum_{k=1}^{n-2}(\theta Q_{r+k+1} - Q_{r+k+2}) \cdot y^{n-(k+1)}.$$

Then we have

$$\Sigma D_{r,n}\Omega_1\Omega_2 = \begin{pmatrix} Q_{r+1} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & q_n & \ddots & & & & \vdots \\ \vdots & 0 & c & \ddots & & & \vdots \\ \vdots & \vdots & d & c & \ddots & & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & d & c \end{pmatrix} = \mathcal{M} \oplus \mathcal{H},$$

where $c = Q_{r+n+1} + Q_{r+1}$, $d = Q_{r+n} + Q_r$, $\mathcal{M} = \text{diag}(Q_{r+1}, q_n)$ is a diagonal matrix, and $\mathcal{M} \oplus \mathcal{H}$ is the direct sum of $\mathcal{M}$ and $\mathcal{H}$.

If we denote $\Omega = \Omega_1\Omega_2$, then we obtain

$$D_{r,n}^{-1} = \Omega(\mathcal{M}^{-1} \oplus \mathcal{H}^{-1})\Sigma.$$

Since the last row elements of the matrix Ω are $(0, 1, \mu_{2,3}, \mu_{2,4}, \ldots, \mu_{2,n})$, the last row elements of the matrix $\Omega(\mathcal{M}^{-1} \oplus \mathcal{H}^{-1})$ are $(0, \frac{1}{q_n}, \nu_{2,3}, \cdots, \nu_{2,n})$, where

$$\nu_{2,3} = \sum_{i=1}^{n-2} \mu_{2,2+i} \cdot \frac{(-d)^{i-1}}{c^i},$$
$$\nu_{2,k} = \sum_{i=1}^{n-(k-1)} \mu_{2,k-1+i} \cdot \frac{(-d)^{i-1}}{c^i}, \ (k = 4, 5, \ldots, n).$$

Hence, the last row elements of the matrix $\Omega(\mathcal{M}^{-1} \oplus \mathcal{H}^{-1})\Sigma$ are $(\omega_1, \omega_2, \ldots, \omega_n)$, where

$$\omega_1 = \frac{1}{q_n}\theta + \nu_{23}, \ \omega_2 = \nu_{2n}, \ \omega_3 = \nu_{2n-1} - 2\nu_{2n},$$
$$\omega_k = \nu_{2,n-k+2} - 2\nu_{2,n-k+3} - \nu_{2,n-k+4}, \ (k = 4, 5, \ldots, n-1),$$
$$\omega_n = \frac{1}{q_n} - 2\nu_{23} - \nu_{24}.$$

By Lemma 1.5, if let $D_{r,n}^{-1} := \text{SCirc}(y_1, y_2, \ldots, y_n)$, then its last row elements $(-y_2, -y_3, \ldots, -y_n, y_1)$, and are given by the following equations:

$$
\begin{aligned}
y_1 &= \omega_n = \frac{1}{q_n} - 2\nu_{2,3} - \nu_{2,4} \\
&= \frac{1}{q_n}[1 - \sum_{i=1}^{n-3}(Q_{r+n+2-i} - \theta Q_{r+n+1-i}) \cdot \frac{(-d)^{i-1}}{c^i} - 2(Q_{r+3} - \theta Q_{r+2})\frac{(-d)^{n-3}}{c^{n-2}}], \\
-y_2 &= \omega_1 = \frac{1}{q_n}\theta + \nu_{2,3} = \frac{1}{q_n}\theta + \frac{1}{q_n}\sum_{i=1}^{n-2}(Q_{r+n+1-i} - Q_{r+n-i})\frac{(-d)^{i-1}}{c^i} \\
-y_3 &= \omega_2 = \nu_{2,n} = \frac{1}{q_n}(Q_{r+3} - \theta Q_{r+2}) \cdot \frac{1}{c}, \\
-y_4 &= \omega_3 = \nu_{2,n-1} - 2\nu_{2,n} = \mu_{2,n-1}\frac{1}{c} + \mu_{2,n}\frac{-d}{c^2} - 2\mu_{2,n}\frac{1}{c} \\
&= \frac{1}{q_n} \cdot \sum_{i=1}^{2}(Q_{r+1+i} - \theta Q_{r+i})\frac{(-d)^{i-1}}{c^i},
\end{aligned}
$$

$$
\begin{aligned}
-y_k &= \nu_{2,n-k+3} - 2\nu_{2,n-k+4} - \nu_{2,n-k+5} \\
&= \frac{1}{q_n}\sum_{i=1}^{k-2}(Q_{r+k+1-i} - \theta Q_{r+k-i}) \cdot \frac{(-d)^{i-1}}{c^i} - \frac{2}{q_n}\sum_{i=1}^{k-3}(Q_{r+k-i} - \theta Q_{r+k-1-i}) \cdot \frac{(-d)^{i-1}}{c^i} \\
&- \frac{1}{q_n}\sum_{i=1}^{k-4}(Q_{r+k-1-i} - \theta Q_{r+k-2-i}) \cdot \frac{(-d)^{i-1}}{c^i} \\
&= \frac{1}{q_n}\sum_{i=1}^{2}(Q_{r+1+i} - \theta Q_{r+i}) \cdot \frac{(-d)^{k-5+i}}{c^{k-4+i}}, \ (k = 5, 6, \ldots, n).
\end{aligned}
$$

Hence, we obtain

$$D_{r,n}^{-1} = \frac{1}{q_n} SCirc(y_1^{'}, y_2^{'}, \ldots, y_n^{'}),$$

where

$$
\begin{aligned}
y_1^{'} &= 1 - \sum_{i=1}^{n-3}(Q_{r+n+2-i} - \theta Q_{r+n+1-i}) \cdot \frac{(-d)^{i-1}}{c^i} - 2(Q_{r+3} - \theta Q_{r+2})\frac{(-d)^{n-3}}{c^{n-2}}, \\
y_2^{'} &= -\theta - \sum_{i=1}^{n-2}(Q_{r+n+1-i} - Q_{r+n-i})\frac{(-d)^{i-1}}{c^i}, \\
y_3^{'} &= -(Q_{r+3} - \theta Q_{r+2}) \cdot \frac{1}{c}, \\
y_4^{'} &= -\sum_{i=1}^{2}(Q_{r+1+i} - \theta Q_{r+i})\frac{(-d)^{i-1}}{c^i}, \\
&\ldots \\
y_k^{'} &= -\sum_{i=1}^{2}(Q_{r+1+i} - \theta Q_{r+i}) \cdot \frac{(-d)^{k-5+i}}{c^{k-4+i}}, (k = 5, 6, \ldots, n).
\end{aligned}
$$

□

3 Norm and Spread of Skew Circulant Matrix with the Pell Lucas Numbers

Theorem 3.1. *Let* $D_{r,n} := \mathrm{SCirc}(Q_{r+1}, Q_{r+2}, \ldots, Q_{r+n})$*. Then three kinds norms of* $D_{r,n}$ *are given by*

$$\|D_{r,n}\|_1 = \|D_{r,n}\|_\infty = \frac{1}{2}(Q_{r+n+1} + Q_{r+n} - Q_{r+1} - Q_r), \tag{3.1}$$

$$\|D_{r,n}\|_F = \sqrt{\frac{n}{2}(Q_{r+n+1}Q_{r+n} - Q_{r+1}Q_r)}. \tag{3.2}$$

Proof. On the basis of the definitions of norms, we have

$$\|D_{r,n}\|_1 = \|D_{r,n}\|_\infty = \sum_{i=1}^{n} Q_{r+i} = \frac{1}{2}(Q_{r+n+1} + Q_{r+n} - Q_{r+1} - Q_r).$$

By (1.2), we have $\sum\limits_{i=0}^{n-1} Q_i^2 = \frac{1}{2}(Q_{n+1}Q_n) + 2$. Hence

$$\begin{aligned}\| D_{r,n} \|_F^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} | a_{ij} |^2 = n\sum_{i=1}^{n} Q_{r+i}^2 \\ &= n(\sum_{i=0}^{r+n} Q_i^2 - \sum_{i=0}^{r} Q_i^2) \\ &= \frac{n}{2}(Q_{r+n+1}Q_{r+n} - Q_{r+1}Q_r).\end{aligned}$$

Thus

$$\|D_{r,n}\|_F = \sqrt{\frac{n}{2}(Q_{r+n+1}Q_{r+n} - Q_{r+1}Q_r)}.$$

□

Theorem 3.2. *Let* $D'_{r,n} = \mathrm{SCirc}(Q_{r+1}, -Q_{r+2}, \ldots, -Q_{r+n-1}, Q_{r+n})$ *be an odd-order alternative skew-circulant matrix, and* $1 = n(mod\ 2)$*. Then*

$$\|D'_{r,n}\|_2 = \sum_{i=1}^{n} Q_{r+i} = \frac{1}{2}(Q_{r+n+1} + Q_{r+n} - Q_{r+1} - Q_r).$$

Proof. We employ [23] and [22] to calculate the spectral norm of $D'_{r,n}$ as follows, for all $j = 0, 1, \ldots, n-1$:

$$\begin{aligned}|\lambda_j(D'_{r,n})| &= \left|\sum_{i=1}^{n}(-1)^{i-1}Q_{r+i}(\omega^j\alpha)^{i-1}\right| \\ &\le \sum_{i=1}^{n} |(-1)^{i-1}Q_{r+i}| \cdot |(\omega^j\alpha)^{i-1}| = \sum_{i=1}^{n} Q_{r+i}.\end{aligned}$$

Since all skew-circulant matrices are normal, we deduce that $\|D'_{r,n}\|_2 = \max\limits_{0\le j\le n-1}\left|\lambda_j(D'_{r,n})\right|$.

If n is odd, then $\sum_{i=1}^{n} Q_{r+i}$ is an eigenvalue of $D'_{r,n}$ as follows

$$\begin{pmatrix} Q_{r+1} & -Q_{r+2} & Q_{r+3} & \cdots & Q_{r+n} \\ -Q_{r+n} & Q_{r+1} & -Q_{r+2} & \vdots & -Q_{r+n-1} \\ Q_{r+n-1} & -Q_{r+n} & Q_{r+1} & \vdots & Q_{r+n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Q_{r+2} & -Q_{r+3} & Q_{r+4} & \cdots & Q_{r+1} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ 1 \end{pmatrix} = \sum_{i=1}^{n} Q_{r+i} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ 1 \end{pmatrix}.$$

Noticing that Lemma 1.2, we claim that $\sum_{i=1}^{n} Q_{r+i}$ is the maximum of $|\lambda_j(D'_{r,n})|$, and we obtain

$$\|D'_{r,n}\|_2 = \sum_{i=1}^{n} Q_{r+i} = \frac{1}{2}(Q_{r+n+1} + Q_{r+n} - Q_{r+1} - Q_r).$$

This completes the proof. □

Theorem 3.3. *Let* $D_{r,n} := \mathrm{SCirc}(Q_{r+1}, Q_{r+2}, ..., Q_{r+n})$. *Then the bounds for the spread of* $D_{r,n}$ *are*

$$s(D_{r,n}) \le \sqrt{n(Q_{r+n}Q_{r+n+1} - Q_{r+1}Q_{r+2})}$$

and

$$s(D_{r,n}) \ge \frac{1}{2(n-1)}|(n-4)Q_{r+n+1} + (n-2)Q_{r+n} + (n+2)Q_{r+2} + nQ_{r+1}|.$$

Proof. We use the result of Theorem 3.1,

$$\| D_{r,n} \|_F^2 = \frac{n}{2}(Q_{r+n+1}Q_{r+n} - Q_rQ_{r+1}),$$

and $\mathrm{tr}D_{r,n} = nQ_{r+1}$. Hence

$$s(D_{r,n}) \le \sqrt{n(Q_{r+n}Q_{r+n+1} - Q_{r+1}Q_{r+2})}.$$

Since

$$\begin{aligned} \sum_{i=1}^{n-1} iQ_i &= \sum_{k=1}^{n-1} \sum_{i=n-k}^{n-1} Q_i = \sum_{k=1}^{n-1}(\sum_{i=0}^{n-1} Q_i - \sum_{i=0}^{n-k-1} Q_i) \\ &= \frac{1}{2}[(n-2)Q_n + (n-1)Q_{n-1} + 2], \end{aligned}$$

it yields that

$$\begin{aligned} \sum_{i \ne j} a_{ij} &= \sum_{k=2}^{n}(n-(k-1))Q_{r+k} - \sum_{k=2}^{n}(k-1)Q_{r+k} \\ &= (n+2)\sum_{k=2}^{n} Q_{r+k} - 2\sum_{k=2}^{n} kQ_{r+k} \\ &= (n+2r+2)\sum_{k=2}^{n} Q_{r+k} - 2\sum_{k=2}^{n}(r+k)Q_{r+k} \\ &= \frac{1}{2}(n+2r+2)(Q_{r+n+1} + Q_{r+n} - Q_{r+2} - Q_{r+1}) \\ &- [(r+n-1)Q_{r+n+1} + (r+n)Q_{r+n} - rQ_{r+2} - (r+1)Q_{r+1}] \\ &= \frac{1}{2}[(4-n)Q_{r+n+1} + (2-n)Q_{r+n} - (n+2)Q_{r+2} - nQ_{r+1}]. \end{aligned}$$

Thus

$$s(D_{r,n}) \geq \frac{1}{2(n-1)}|(n-4)Q_{r+n+1} + (n-2)Q_{r+n} + (n+2)Q_{r+2} + nQ_{r+1}|.$$

□

4 Determinant and Inverse of Skew Left Circulant Matrix with the Pell-Lucas Numbers

In this section, let $D'_{r,n} := \mathrm{SLCirc}(Q_{r+1}, Q_{r+2}, \ldots, Q_{r+n})$, where Q_i be the i-th Pell-Lucas number for $i = r+1, \ldots, r+n$. Firstly, we give a determinant explicit formula for the matrix $D'_{r,n}$. Afterwards, we prove that $D'_{r,n}$ is an invertible matrix for any positive integer n, and then we find the inverse of the matrix $D'_{r,n}$. By Lemma 1.2–1.3, Theorem 3.1–3.3, we can obtain the following theorem.

Theorem 4.1. *Let $D'_{r,n} := \mathrm{SLCirc}(Q_{r+1}, Q_{r+2}, \ldots, Q_{r+n})$. Then we have*

(i). *$D'_{r,n}$ is invertible for any positive integer n,*

(ii). $$\det D'_{r,n} = (-1)^{\frac{n(n-1)}{2}} Q_{r+1} \cdot [Q_{r+1} + \theta Q_{r+n} + \sum_{k=1}^{n-2}(\theta Q_{r+k+1} - Q_{r+k+2}) \cdot y^{n-(k+1)}] \cdot (Q_{r+1} + Q_{r+n+1})^{n-2},$$

(iii). $$D'^{-1}_{r,n} = \frac{1}{q_n}(y_1^{''}, y_2^{''}, \ldots, y_n^{''}),$$

where

$$y_1^{''} = 1 - \sum_{i=1}^{n-3}(Q_{r+n+2-i} - \theta Q_{r+n+1-i}) \cdot \frac{(-d)^{i-1}}{c^i} - 2(Q_{r+3} - \theta Q_{r+2}) \cdot \frac{(-d)^{n-3}}{c^{n-2}},$$

$$y_2^{''} = \sum_{i=1}^{2}(Q_{r+1+i} - \theta Q_{r+i}) \cdot \frac{(-d)^{n-5+i}}{c^{n-4+i}},$$

$$y_k^{''} = \sum_{i=1}^{2}(Q_{r+1+i} - \theta Q_{r+i}) \cdot \frac{(-d)^{n-3-k+i}}{c^{n-2-k+i}}, (k = 3, 4, \ldots, n-3),$$

$$y_{n-2}^{''} = \sum_{i=1}^{2}(Q_{r+1+i} - \theta Q_{r+i}) \cdot \frac{(-d)^{i-1}}{c^i},$$

$$y_{n-1}^{''} = (Q_{r+3} - \theta Q_{r+2}) \cdot \frac{1}{c},$$

$$y_n^{''} = \theta + \sum_{i=1}^{n-2}(Q_{r+n+1-i} - Q_{r+n-i}) \cdot \frac{(-d)^{i-1}}{c^i}.$$

5 Norm and Spread of Skew Left Circulant Matrix with the Pell Lucas Numbers

Theorem 5.1. *Let $D'_{r,n} := \mathrm{SLCirc}(Q_{r+1}, Q_{r+2}, ..., Q_{r+n})$ $(n > 2)$. Then three kinds norms of $D'_{r,n}$ are given by*

$$\|D'_{r,n}\|_1 = \|D'_{r,n}\|_\infty = \frac{1}{2}(Q_{r+n+1} + Q_{r+n} - Q_{r+1} - Q_r),$$

$$\|D'_{r,n}\|_F = \sqrt{\frac{n}{2}(Q_{r+n+1}Q_{r+n} - Q_rQ_{r+1})}. \tag{5.1}$$

Theorem 5.2. *Let* $D''_{r,n} := \text{SLCirc}(Q_{r+1}, -Q_{r+2}, ..., -Q_{r+n-1}, Q_{r+n})$ *and* n *be odd. Then*

$$\|D''_{r,n}\|_2 = \sum_{i=1}^{n} Q_{r+i} = \frac{1}{2}(Q_{r+n+1} + Q_{r+n} - Q_{r+1} - Q_r).$$

Proof. According to Lemma 1.2, we have

$$\lambda_j(D''_{r,n}) = \pm|\sum_{i=1}^{n}(-1)^{i-1}Q_{r+i}\omega^{(j-\frac{1}{2})(k-1)}|,\ (j = 1, 2, \dots, \frac{n-1}{2}),$$

and

$$\lambda_{\frac{n+1}{2}}(D''_{r,n}) = \sum_{i=1}^{n} Q_{r+i}, \tag{5.2}$$

thus

$$\begin{aligned} |\lambda_j(D''_{r,n})| &\le \sum_{i=1}^{n} |(-1)^{i-1}Q_{r+i}(-1)^{i-1}| \\ &= \sum_{i=1}^{n} Q_{r+i},\ (j = 1, 2, \dots, \frac{n+1}{2}). \end{aligned} \tag{5.3}$$

By Equation (5.2) and (5.3), we have

$$\max_{0\le i\le \frac{n+1}{2}} \left|\lambda_i(D''_{r,n})\right| = \sum_{i=1}^{n} Q_{r+i}. \tag{5.4}$$

By Lemma 1.6, Equation (1.1) and Equation (5.3), we obtain

$$\|D''_{r,n}\|_2 = \sum_{i=1}^{n} Q_{r+i} = \frac{1}{2}(Q_{r+n} + Q_{r+n+1} - Q_{r+1} - Q_r).$$

□

Theorem 5.3. *Let* $D'_{r,n} := \text{SLCirc}(Q_{r+1}, Q_{r+2}, ..., Q_{r+n})$ *. Then the bound for the spread of* $D'_{r,n}$ *satisfies:*

(i) *if* n *is odd, then*

$$2Q_{r+n} \le s(D'_{r,n}) \le \sqrt{nM - \frac{1}{2n}N^2},$$

where

$$\begin{aligned} M &= Q_{r+n}Q_{r+n+1} - Q_rQ_{r+1}, \\ N &= Q_{r+n} + Q_{r+n-1} + Q_{r+1} - Q_r; \end{aligned}$$

(ii) *if* n *is even, then*

$$2Q_{r+n} \le s(D'_{r,n}) \le \sqrt{n(Q_{r+n}Q_{r+n+1} - Q_rQ_{r+1})}.$$

Proof. Since $D'_{r,n}$ is a symmetric matrix, by Lemma 1.7, $s(D'_{r,n}) \geq 2\max_{i\neq j}|a_{ij}| = 2Q_{r+n}$, in addition,

(i) if n is odd,

$$\begin{aligned}\mathrm{tr}(C'_{r,n}) &= Q_{r+1} - Q_{r+2} + Q_{r+3} - \cdots + Q_{r+n} \\ &= Q_{r+1} + Q_{r+1} + Q_{r+2} + \cdots + Q_{r+n-1} \\ &= Q_{r+1} + \sum_{i=1}^{n-1} Q_{r+i}.\end{aligned}$$

By Equation (1.1) we have

$$\mathrm{tr}(D'_{r,n}) = \frac{1}{2}(Q_{r+n} + Q_{r+n-1} + Q_{r+1} - Q_r). \tag{5.5}$$

Let

$$\begin{aligned}M &= Q_{r+n}Q_{r+n+1} - Q_rQ_{r+1}, \\ N &= Q_{r+n} + Q_{r+n-1} + Q_{r+1} - Q_r.\end{aligned}$$

By Equation (1.10), (5.1) and (5.5), we obtain

$$2Q_{r+n} \leq s(C'_{r,n}) \leq \sqrt{nM - \frac{1}{2n}N^2}.$$

(ii) If n is even,

$$\mathrm{tr}(D'_{r,n}) = Q_{r+1} - Q_{r+1} + Q_{r+3} - Q_{r+3} + \cdots + Q_{r+n-1} - Q_{r+n-1} = 0. \tag{5.6}$$

By Equation (1.10), (5.1) and (5.6), we obtain

$$2Q_{r+n} \leq s(D'_{r,n}) \leq \sqrt{n(Q_{r+n}Q_{r+n+1} - Q_rQ_{r+1})}.$$

□

6 Conclusion

The determinants and inverses of skew circulant and skew left circulant matrices with Pell and Pell-Lucas numbers is discussed in this paper, and four kinds norms and bounds for the spread of these matrices are given, respectively.

Acknowledgement

This project is supported by the National Natural Science Foundation of China (Grant No. 11671187), Natural Science Foundation of Shandong Province (Grant No. ZR2016AM14) and the AMEP of Linyi University, China.

Competing Interests

Authors have declared that no competing interests exist.

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